HOMOMORPHISMS AND DERIVATIONS ON WEIGHTED CONVOLUTION ALGEBRAS

by

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PREFACE

The material presented in this thesis is claimed as original with the exception of those sections where specific mention is made to the contrary.

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ABSTRACT

This thesis consists of two separate and distinct parts.

Part One is concerned with the problem of characterizing of homomorphisms and derivations on the algebra $L^1(\omega)$.

Chapter 1.1 is on general properties of $L^1(\omega)$. In this chapter we prove that every continuous endomorphism of $L^1(\omega)$ has an extension to a continuous endomorphism of $M(\omega)$.

In Chapter 1.2 we characterize isomorphisms from one semi-simple algebra $L^1(\omega_1)$ onto another semi-simple algebra $L^1(\omega_2)$. In this chapter we also study the endomorphisms of $L^1(R^+)$.

In Chapter 1.3 we characterize the isometric isomorphisms of a radical $L^1(\omega)$. We also find a necessary and sufficient condition for two radical algebras $L^1(\omega_1)$ and $L^1(\omega_2)$ to be isometrically isomorphic.

Chapter 1.4 is on derivations of $L^1(\omega)$. In this chapter we characterize derivations on a radical $L^1(\omega)$ and we find necessary and sufficient conditions on ω for the existence of non-zero derivations.

Part Two is on isometric representations of the algebras $M(G)$. The main results of this part are in Chapter 2.2. In this chapter we prove that there is an isometric isomorphism from $M(G)$ into $BB(H)$ and the algebra $L^1(G)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space.

CONTENTS

INTRODUCTION

This thesis consists of two parts. Part One is on homomorphisms and derivations of the weighted convolution algebras $L^1(\omega)$. Part Two is on isometric representations of the measure algebra $M(G)$ on $B(H)$, where G is a locally compact Hausdorff topological group.

A weight ω on the non-negative real numbers is a continuous positive function such that $\omega(s + t) \leq \omega(s)\omega(t)$ for all non-negative s and t and $\omega(0) = 1$. The weighted convolution algebra $L^1(\omega)$ corresponding to ω is the algebra of all Lebesgue measurable functions under the usual pointwise addition, scalar multiplication, the convolution product, and norm,

$$||f|| = \int_0^\infty |f(t)|\omega(t)\ dt \qquad (f \in L^1(\omega))$$

We let $M(\omega)$ be the convolution Banach algebra of complex regular Borel measures, under the usual addition and scalar multiplication of measures and norm defined by,

$$||\mu|| = \int_0^\infty \omega(t)\ d|\mu|(t) \qquad (\mu \in M(\omega))$$

where $|\mu|$ is the total variation of the measure μ.

A homomorphism from one Banach algebra A into another Banach algebra B is a linear mapping θ, with the property,

$$\theta(ab) = \theta(a)\theta(b) \qquad (a, b \in A)$$

If θ is from A into A we call it an endomorphism and if it is one-to-one it is called a monomorphism. An automorphims on A is a monomorphism from A onto A . A homomorphism θ is said to be isometric if $||\theta(a)|| = ||a||$ for every $a \in A$. A derivation on an algebra A is a linear mapping D which satisfies

$$D(ab) = D(a)b + aD(b) \qquad (a, b \in A)$$

A great deal of research has been done on the characterization of homomorphisms and derivations from one Banach algebra into another Banach algebra, when the two Banach algebras are of a particular type. Here we give a brief survey of this subject. The Harmanic Analysis part of the subject began with the work of Wendel who characterized the isometric isomorphisms of the group algebras [cf.37]. After Wendel's paper Harmonic Analysts worked on the characterization of homomorphisms from one group algebra into another group algebra. Helson [cf.15], Beurling and Helson [cf.2], Leibenson [cf.22], Kahane [cf.19] and Rudin [cf.29, 30] have given partial solutions and the general result is given by P.J. Cohen [cf.7] when the two underlying groups are commutative. Some work has also been done on the homomorphisms of the algebra of absolutely convergent power series [cf.25]. A great deal of work has been done on the derivations and automorphisms of C*-algebras and Von Neumann algebras [cf.31].

Perhaps the most general result about the characterization of derivations on commutative semi-simple Banach algebras is due to Johnson [cf.18], which states that in a commutative semi-simple

Banach algebra every derivation is zero. However, when a commutative Banach algebra A is not semi-simple, especially when it is radical, several questions about the nature of derivations and homomorphisms can be asked. The radical Banach algebra $L^1(0, 1)$ with convolution product, is one example. Kamowitz and Scheinberg have studied the derivations and automorphisms of the algebra $L^1(0, 1)$ [cf.20]. Unaware of the fact that every derivation on $L^1(0, 1)$ is continuous, they have characterized all continuous derivations on $L^1(0, 1)$. Sinclair and Jewell [cf.17] later, amongst other things, proved that every derivation on $L^1(0, 1)$ is continuous and this combined with the result of Scheinberg and Kamowitz gives a characterization of the derivations on $L^1(0, 1)$. Diamond [cf.11] has characterized all the derivations on convolution algebras of complex measures on non-negative half line which are of finite variation on every compacta.

In Part One of this thesis we have tried to solve problems concerning homomorphisms and derivations on the algebras $L^1(\omega)$. These algebras are either semi-simple or radical Banach algebras according as $\lim_{t\to\infty} \omega(t)^{1/t}$ is different from 0 or equal to 0. Recently there has been some interest in radical algebras $L^1(\omega)$ both for their connection with some problems of the automatic continuity [cf.10] and their closed ideal structure [cf.9].

We have divided Part One into four chapters denoted by 1.1 - 1.4. Chapter 1.1 is on general properties of the algebras $L^1(\omega)$. The main result of this chapter is proposition 1.1.12, which is on extension of a continuous endomorphism of $L^1(\omega)$ to a continuous endormorphism of $M(\omega)$, in both semi-simple and radical cases.

Chapter 1.2 is on semi-simple $L^1(\omega)$. In this chapter we have characterized the isomorphisms from one semi-simple $L^1(\omega_1)$ onto another semi-simple $L^1(\omega_2)$. We have studied the algebra $L^1(R^+)$ which corresponds to $\omega(t) = 1$ in more detail. We have shown that every endomorphism of $L^1(R^+)$ is a monomorphism and have given a formula for all endormorphisms of $L^1(R^+)$. In chapter 1.3 we have characterized the isometric isomorphisms of a radical $L^1(\omega)$; there are very few of them. If θ is an isometric isomorphism of $L^1(\omega)$ then there is a real number α such that

$$(\theta f)(x) = e^{i\alpha x} f(x) \qquad (x \geq 0,\ f \in L^1(\omega))$$

By using the methods of Chapter 1.3 we can prove that if $L^1(\omega_1)$ and $L^1(\omega_2)$ are two radical weighted algebras, then $L^1(\omega_1)$ is isometrically isomorphic to $L^1(\omega_2)$ if and only if there exist $a > 0$ and $b > 0$ such that

$$\frac{\omega_1(x)}{\omega_2(ax)} = b^x$$

for every non-negative x. Chapter 1.4 is on derivations of a radical $L^1(\omega)$. If D is a derivation on a radical $L^1(\omega)$, then there is a locally finite regular Borel measure μ such that,

$$\text{(I)} \qquad Df = tf*\mu \qquad (f \in L^1(\omega))$$

with $$\text{(II)} \qquad \sup_{t>0} \frac{t}{\omega(t)} \int_0^\infty \omega(t+s)\, d|\mu|(s) < \infty$$

where $\qquad (tf)(x) = xf(x)$ for all non-negative x.

We have found the norm of the derivation D in terms of μ and this is given by,

$$||D|| = \sup_{t>0} \frac{t}{\omega(t)} \int_0^\infty \omega(t+s)\, d\, |\mu|\ (s) .$$

Conversely a map D defined on $L^1(\omega)$ by (I) which satisfies condition (II) is a derivation on $L^1(\omega)$. A necessary and sufficient condition for the existence of a non-zero derivation on the algebra $L^1(\omega)$ is the existence of a positive number b, such that

$$\sup_{t>0} t \frac{\omega(t+b)}{\omega(t)} < \infty .$$

To be less formal, if ω tends to zero fast as $t \to \infty$ then there exist non-zero derivations and if it tends to zero slowly as $t \to \infty$, there is no non-zero derivation.

Given two radical algebras $L^1(\omega_1)$ and $L^1(\omega_2)$ we have shown that a necessary and sufficient condition for $L^1(\omega_2)$ to be a two-sided Banach $L^1(\omega_1)$-module, with the module product the convolution product, is that,

$$\sup_{t>0} \frac{\omega_2(t)}{\omega_1(t)} < \infty .$$

A derivation from $L^1(\omega_1)$ into $L^1(\omega_2)$ is given by

$$Df = tf*\mu \qquad (f \in L^1(\omega_1))$$

where μ is a locally finite regular Borel measure which satisfies

$$||D|| = \sup_{t>0} \frac{t}{\omega_1(t)} \int_0^\infty \omega_2(t+s)\, d|\mu|(s) < \infty$$

and a necessary and sufficient condition for the existence of a non-zero derivation is the existence of a positive number b such that

$$\sup_{t>0} \frac{t}{\omega_1(t)} \omega_2(t+b) < \infty \quad .$$

In a commutative Banach algebra, the exponential of a continuous derivation is an automorphism [cf.3]. Therefore, theoretically, we know the subgroup of the group of the automorphisms of the algebra $L^1(\omega)$, whose elements are $\exp D$. Perhaps this can be used to find a general formula for the automorphisms of the algebras $L^1(\omega)$, in the radical case.

Part Two of this thesis grew out of an attempt at finding an isometric representation of the extremal algebra $Ea[-1, 1]$, [cf.8] on $B(H)$, and led to an isometric representation of the measure algebra $M(G)$ of a locally compact Hausdorff group on $B(H)$. In this part we have also shown that the group algebra $L^1(G)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space. This combined with the fact that $B(H)$ has an isometric embedding in $BB(H)$, by the left regular representation, shows that the algebra $BB(H)$ is a large algebra in comparison with the algebra $B(H)$. However, despite the fact that $Ea[-1, 1]$ is a quotient of the group algebra of real numbers with discrete topology by a closed ideal [cf.35], we have not been able to find an isometric isomorphism from $Ea[-1, 1]$ into $BB(H)$, this and a more general question of whether $BB(H)$ contains the quotients of the algebra $L^1(G)$ by its closed ideals remains open.

We have divided Part Two into two chapters which are denoted by 2.1 and 2.2. Chapter 2.1 is a necessary background for Chapter 2.2 and the material in this chapter is not new. The main results of Chapter 2.2 are the isometric representation of $M(G)$ on $B(H)$ [Theorem 2.2.11] and the non-representability (isometric) of $L^1(G)$ on a Hilbert space [Theorem 2.2.22].

In both Parts One and Two knowledge of basic functional analysis is assumed [cf. 12]. In Part Two some more specialized results from Harmonic Analysis, which can be found in [16], and from the theory of Numerical ranges in Banach algebras [cf. 4] as well as unitary representations of groups [cf. 1] are quoted without proof. In Part One we have assumed familiarity with the fact that the algebra $L^1(R)$ is semi-simple, and that if $\hat{f}$ is the Fourier transform of a function $f \in L^1(R)$, then $\lim_{x\to\infty} \hat{f}(x) = 0$ [Riemann-Lebesgue lemma]. We have used the following notation and definition, $\underset{\sim}{R}$ denotes the real numbers, and $\underset{\sim}{C}$ the complex numbers, R^+ denotes the non-negative real numbers. We denote the rational numbers by Q and non-negative rational numbers by Q^+. If X is a locally compact Hausdorff topological space, we denote the σ-algebra of Borel subsets of X by B and if f is a complex valued function defined on X, then the support of f is the closure of the set $\{x : f(x) \neq 0\}$. If μ is a Borel measure on X, then the support of μ is the complement of the union of open sets which are of μ-measure zero. We denote the space of all complex valued continuous functions on X which are with compact support by $C_c(X)$ and the space of all complex valued continuous functions which vanish at infinity by $C_0(X)$. We denote the set of non-negative continuous functions with compact support by $C_c^+(X)$. Finally, all the vector spaces in this thesis are over the field of complex numbers.

PART ONE

CHAPTER 1.1 THE ALGEBRAS $L^1(\omega)$ and $M(\omega)$

1.1.1 Definition. Let ω be a continuous and positive function on R^+, $\omega(0) = 1$, and let ω be submultiplicative, i.e.

$$(1) \qquad \omega(s+t) \leq \omega(s)\omega(t) \qquad (s, t \in R^+)$$

Then we call ω a weight function or simply a weight on R^+.

Let $L^1(\omega)$ denote the set of all Lebesgue measurable functions defined on R^+, such that for every $f \in L^1(\omega)$ we have

$$(2) \qquad \int_0^\infty |f(t)|\omega(t)dt < \infty$$

As usual by a function f we mean the class of all functions which are equal to f almost everywhere. The space $L^1(\omega)$ with the usual pointwise addition of functions, scalar multiplication, convolution product, and norm defined as below is a Banach algebra:

$$(3) \qquad \begin{aligned} (f+g)(x) &= f(x) + g(x) && (f, g \in L^1(\omega),\ x \in R^+) \\ (\lambda f)(x) &= \lambda f(x) && (f \in L^1(\omega),\ \lambda \in \mathbb{C},\ x \in R^+) \\ (f*g)(x) &= \int_0^x f(x-y)g(y)dy && (f, g \in L^1(\omega),\ a.e.\ x \in R^+) \\ \|f\| &= \int_0^\infty |f(t)|\omega(t)dt && (f \in L^1(\omega)) \end{aligned}$$

For every weight ω let $M(\omega)$ denote the set of all complex regular Borel measures μ such that

$$(4) \qquad \int_0^\infty \omega(t)d|\mu|(t) < \infty$$

where $|\mu|$ is the total variation of μ. With the addition of two measures, scalar multiplication and norm defined as below $M(\omega)$ is a Banach space,

$$(5) \qquad (\mu+\nu)(E) = \mu(E)+\nu(E) \qquad (\mu,\ \nu \in M(\omega),\ E \in B)$$

$$(\lambda\mu)(E) = \lambda\mu(E) \qquad (\lambda \in \underset{\sim}{C},\ E \in B)$$

$$||\mu|| = \int_0^\infty \omega(t)\ d|\mu|(t) \qquad (\mu \in M(\omega))$$

We denote by $C_0(\omega)$ the space of all complex valued functions f on R^+ such that $f/\omega \in C_0(R^+)$ [continuous functions on R^+ which vanish at infinity). The space $C_0(\omega)$ with addition and scalar multiplication as in (3) and norm defined by

$$(6) \qquad ||f|| = ||f/\omega||_\infty \qquad (f \in C_0(\omega))$$

is a Banach space. The Banach space $M(\omega)$ can be identified with the dual of $C_0(\omega)$ by the pairing

$$(7) \qquad \langle\mu,\ \psi\rangle = \int_0^\infty \psi(x)d\mu(x) \qquad (\mu \in M(\omega),\ \psi \in C_0(\omega))$$

Given $\mu,\ \nu \in M(\omega)$, let $\mu*\nu$ be a measure in $M(\omega)$ defined by

$$(8) \qquad \int_0^\infty \psi(t)d(\mu*\nu)(t) = \int_0^\infty\int_0^\infty \psi(s+t)d\mu(s)d\nu(t) \qquad (\psi \in C_0(\omega))$$

The Banach space $M(\omega)$ with product $*$ is a Banach algebra and the algebra $L^1(\omega)$ can be regarded as a closed subalgebra of $M(\omega)$. Indeed, for every $f \in L^1(\omega)$, we let μ be a measure in $M(\omega)$ defined by

$$(9) \qquad d\mu(x) = f(x)dx$$

where dx denotes the Lebesgue measure on R^+. This is an isometric embedding of $L^1(\omega)$ in $M(\omega)$, in fact we have:

Lemma 1.1.2 $L^1(\omega)$ is a closed ideal in $M(\omega)$.

Proof. Let $f \in L^1(\omega)$, $\mu \in M(\omega)$, then from 1.1.1 (8) it follows for every $\psi \in C_0(\omega)$

$$(1) \qquad \int_0^\infty \psi(t)\,d(\mu*f)(t) = \int_0^\infty \int_0^\infty \psi(s+t)\,d\mu(s)f(t)\,dt$$

$$= \int_0^\infty \psi(x) \int_0^x f(x-s)\,d\mu(s)\,dx$$

The last equality in (1) follows by considering the function

$$h(x, s) = \begin{cases} f(x-s) & s < x \\ 0 & \text{elsewhere} \end{cases}$$

Then the last integral in (1) is equal to $\int_0^\infty \psi(x) \int_0^\infty h(x, s)\,d\mu(s)\,dx$.

Now, by Fubini's theorem we have

$$\int_0^\infty \psi(x) \int_0^\infty h(x, s)\,d\mu(s)\,dx = \int_0^\infty \int_0^\infty \psi(x)h(x, s)\,dx\,d\mu(s)$$

$$= \int_0^\infty \int_s^\infty \psi(x)\, f(x-s)\,dx\,d\mu(s)$$

A change of variable $x - s = t$, gives the equality in (1).
The function h defined by $h(x) = \int_0^x f(x-y)\,d\mu(y)$ is in $L^1(\omega)$ and (1) shows that $d(\mu*f) = h\,dx$. Thus $\mu*f \in L^1(\omega)$, and $L^1(\omega)$ is an ideal in $M(\omega)$.

We give some more definitions and notation which we will need in this chapter. On $M(\omega)$ we consider three topologies other than the norm topology and these are:

(a) The weak topology $\sigma = \sigma(M(\omega), C_0(\omega))$.

(b) The strong operator topology denoted by so. This topology in terms of convergence of nets is defined as follows:

a net $\{\mu_\lambda : \mu_\lambda \in M(\omega), \lambda \in \Lambda\}$ tends to a measure μ if and only if $\mu_\lambda * f$ tends to $\mu * f$ in norm for every $f \in L^1(\omega)$.

(c) The bounded strong operator topology denoted by bso . A base of open neighbourhoods of 0 for this topology consists of all sets of the form $X \cap Y$ where X ranges over a base of so open neighbourhoods of 0 and Y is a fixed open norm bounded neighbourhood of 0 . Given a subset $S \subset R^+$, we let $E_S = \left\{\frac{1}{\omega(x)} \delta_x : x \in S\right\}$. If τ is any topology on $M(\omega)$, then $[E_S, \tau]$ will denote E_S with the induced topology τ . If $\{\mu_\lambda : \lambda \in \Lambda\}$ is a net in $M(\omega)$ and τ is a topology on $M(\omega)$, then $\mu_\lambda \xrightarrow{\tau} \mu$ and $\lim\limits_{\tau} \mu_\lambda = \mu$ will mean that $\{\mu_\lambda : \lambda \in \Lambda\}$ tends to μ in the topology τ .

The algebras $L^1(\omega)$ are either semi-simple or radical Banach algebras. There is a necessary and sufficient condition on ω which guarantees when $L^1(\omega)$ is semi-simple or radical. In the semi-simple case we can identify the maximal ideal space of $L^1(\omega)$ with a half-plane in the complex plane. First we need the following lemma [cf.3].

1.1.3 Lemma. The $\lim\limits_{t\to\infty} -\frac{1}{t}\log \omega(t)$ either exists or is ∞ and in each case it is equal to $\sup\limits_{t>0} -\frac{1}{t}\log \omega(t)$.

Proof. Let $b = \sup_{t>0} -\frac{1}{t} \log \omega(t)$. Let $d < b$, then there is an $a > 0$ such that

(1) $$d < -\frac{1}{a} \log \omega(a)$$

suppose now that $x = (n+1)a + c$ where $0 \le c \le a$. Then

(2) $$b \ge -\frac{\log \omega(x)}{x} = \frac{-\log \omega(na + a + c)}{(n+1)a + c}$$

$$\ge \frac{-n \log \omega(a) - \log \omega(a+c)}{(n+1)a + c} \ge \frac{na}{(n+1)a + c}\, d - \frac{M}{(n+1)a + c}$$

where M denotes the maximum of $\log \omega(x)$ over the interval $[a, 2a]$. As $n \to \infty$ the last number tends to d and the result follows.

1.1.4 Definition. For every $\lambda \ge 0$ and $f \in L^1(\omega)$, we let the shift of f by λ, $S_\lambda f$, be the function in $L^1(\omega)$ defined by

$$(S_\lambda f)(x) = \begin{cases} 0 & x \le \lambda \\ f(x-\lambda) & \lambda \le x \end{cases}$$

For each $\lambda \ge 0$, S_λ is a linear operator on $L^1(\omega)$ and

$$\|S_\lambda f\| = \int_\lambda^\infty |f(x-\lambda)|\omega(x)\,dx = \int_0^\infty |f(x)|\omega(x+\lambda)\,dx$$

$$\le \omega(\lambda) \int_0^\infty |f(x)|\omega(x)\,dx$$

Thus, $\|S_\lambda f\| \le \omega(\lambda)\|f\|$, S_λ is bounded and $\|S_\lambda\| \le \omega(\lambda)$

1.1.5 Lemma. For every $f \in L^1(\omega)$, the map $\lambda \to S_\lambda f$ is continuous.

Proof. First let f be a continuous function with compact support. Then, for $\lambda_0 < \lambda$ we have

$$||S_\lambda f - S_{\lambda 0} f|| = \int_{\lambda_0}^{\lambda} |f(x - \lambda_0)|\omega(x)dx + \int_{\lambda}^{\infty} |f(x - \lambda) - f(x - \lambda_0)| \omega(x)dx \to 0$$

as $\lambda \downarrow \lambda_0$. A similar argument with $\lambda \uparrow \lambda_0$ shows that the map $\lambda \to S_\lambda f$ is in this case continuous. For a general $f \in L^1(\omega)$, given $\epsilon > 0$, let $f_0 \in C_c(R^+)$ be such that $||f - f_0|| < \epsilon$. Then

$$||S_\lambda f - S_{\lambda 0} f|| \leq ||(S_\lambda - S_{\lambda_0})(f - f_0)|| + ||(S_\lambda - S_{\lambda 0})f_0||$$

$$\leq (||S_\lambda|| + ||S_{\lambda_0}||)||f - f_0|| + ||(S_\lambda - S_{\lambda 0})f_0||$$

$$\leq [\omega(\lambda) + \omega(\lambda_0)]\epsilon + ||(S_\lambda - S_{\lambda 0})f_0||$$

Since $\omega(\lambda)$ is continuous at λ_0 we get the result.

The proofs of the following lemma and theorem are from [13].

1.1.6 Lemma. A closed ideal of the algebra $L^1(\omega)$ containing the function $f \in L^1(\omega)$ also contains all its 'shifts' $S_\lambda f$ $(\lambda > 0)$, and we have,

$$(1) \qquad S_\lambda f = \lim_{h\to 0^+} \left\{ f * \frac{\chi[0, \lambda+h] - \chi[0, \lambda]}{h} \right\}$$

where the limit is to be understood in the sense of convergence in norm.

Proof. The functions $\frac{\chi[0, \lambda+h] - \chi[0, \lambda]}{h}$ are bounded in norm when h is in a bounded neighbourhood of 0. Hence it follows that it is sufficient to prove the limit relation (1) for the functions $f = \chi_{[a, b]}$ $(0 \leq a < b)$ which are generators of $L^1(\omega)$.

Since $\chi_{[a,b]} = \chi_{[0,b]} - \chi_{[0,a]}$ it suffices that we prove the limit relation (1) for $f = \chi_{[0,b]}$. We prove (1) for $\lambda < b$. Let h be as small as $\lambda + h < b$. We have,

$$(S_\lambda f)(x) = \begin{cases} 0 & x < \lambda \\ 1 & \lambda \le x \le b + \lambda \\ 0 & b + \lambda < x \end{cases}$$

$$\left(f * \frac{\chi_{[0,\ \lambda+h]} - \chi_{[0,\ \lambda]}}{h}\right)(x) = \begin{cases} 0 & x < \lambda \\ \frac{x-\lambda}{h} & \lambda \leqq x < \lambda + h \\ 1 & \lambda+h \leqq x \leqq b + \lambda \\ \frac{b + h + \quad - x}{h} & b + \lambda < x \leqq b + \lambda + h \\ 0 & b + \lambda + h < x \end{cases}$$

Therefore,

$$\left\| S_\lambda f - f * \frac{\chi_{[0,\ \lambda+h]} - \chi_{[0,\ \lambda]}}{h} \right\|$$

$$= \int_\lambda^{\lambda+h} \left(1 - \frac{x-\lambda}{h}\right)\omega(x)\,dx + \int_{b+\lambda}^{b+\lambda+h} \frac{b+h+\lambda-x}{h}\,\omega(x)\,dx$$

and each of these integrals tend to zero as $h \to 0+$. A similar computation with $b \le \lambda$ proves the lemma.

1.1.7 Theorem. If $\lim\limits_{t\to\infty} -\frac{1}{t}\log\omega(t) = \alpha < \infty$, then $L^1(\omega)$ is semi-simple and its maximal ideal space can be identified with the half-plane $H_\alpha = \{z : \mathrm{Re}z \ge \alpha\}$, where for each $z \in H_\alpha$ there corresponds a character Ω_z with,

$$\Omega_z(f) = \int_0^\infty f(t)e^{-zt}dt \qquad (f \in L^1(\omega))$$

and every character arises in this way.

Proof. Let Ω be a character of $L^1(\omega)$. It is easy to see that for every $\lambda > 0$, $f \in L^1(\omega)$, $S_\lambda f \in L^1(\omega)$. Since Ω is not identically zero, there is $f \in L^1(\omega)$, with $\Omega(f) \neq 0$. By lemma (1.1.5) $S_\lambda f$ is a continuous function of λ (in norm). The application of Ω to both sides of (1) in Lemma 1.1.6 shows that,

$$(1) \quad \lim_{h\to 0^+} \Omega\left\{\frac{\chi_{[0,\ \lambda+h]} - \chi_{[0,\ \lambda]}}{h}\right\} = \frac{d}{d\lambda}\Omega(\chi_{[0,\ \lambda]}) = \phi(\lambda)$$

exists for all $\lambda \geq 0$, where

$$(2) \qquad \phi(\lambda) = \frac{\Omega(S_\lambda f)}{\Omega(f)}$$

Furthermore, since for $\lambda, \mu > 0$

$$(3) \qquad (S_{\lambda+\mu}f)*f = (S_\lambda f)*(S_\mu f)$$

we have

$$(4) \qquad \Omega(S_{\lambda+\mu}f)\ \Omega(f) = \Omega(S_\lambda f)\ \Omega(S_\mu f)$$

Dividing by $[\Omega(f)]^2$ and bearing in mind formula (2) we obtain

$$(5) \qquad \phi(\lambda + \mu) = \phi(\lambda)\phi(\mu) \qquad (\lambda, \mu \in R^+)$$

By (2) $\phi(\lambda)$ is a continuous function of λ and since

$$\lim_{h\to 0^+} \left\|\frac{\chi_{[0,\ \lambda+h]} - \chi_{[0,\ \lambda]}}{h}\right\| = \omega(\lambda)$$

from (1) we obtain (6) $|\phi(\lambda)| \leq \omega(\lambda)$.

From (5) and the continuity of the function $\phi(\lambda)$ it follows that

(7) $$\phi(t) = \exp(-zt)$$

where $z = \sigma + i\tau$ is a fixed complex number. The inequality (6) shows that

(8) $$\exp(-\sigma t) \leq \omega(t)$$

for all $t \in R^{+}$, or equivalently

$$-\frac{1}{t}\log \omega(t) \leq \sigma$$

By taking the limit of both sides as $t \to \infty$ we obtain

$$\alpha \leq \sigma$$

Now let $0 \leq a < b$, by integrating both sides of (1) in $[a, b]$ and observing that $\Omega(\chi_{[o,\ b]}) - \Omega(\chi_{[0,\ a]}) = \Omega(\chi_{[a,\ b]})$ we obtain

(9) $$\Omega(\chi_{[a,\ b]}) = \int_a^b e^{-zt}dt = \int_0^\infty \chi_{[a,\ b]}(t)e^{-zt}dt$$

Since the linear span of the functions $\chi_{[a,b]}$ for $0 \leq a < b$ are dense in $L^1(\omega)$. The formula (9) holds for $\chi_{[a,b]}$ replaced by $f \in L^1(\omega)$ and we have

(10) $$\Omega(f) = \int_0^\infty f(t)e^{-zt}dt \qquad (f \in L^1(\omega))$$

Conversely, it is easy to verify that for every $z \in H_\alpha$, the mapping

$$f \to \int_0^\infty f(t)e^{-zt}dt$$

defines a character on $L^1(\omega)$. From (10) it is easy to see that a net Ω_{z_α} of characters tends to a character Ω_z in the topology $\sigma(L^1(\omega), L^1(\omega)^*)$ if and only if $z_\alpha \xrightarrow{|\cdot|} z$.

To prove $L^1(\omega)$ is semi-simple let for $f \in L^1(\omega)$

$$\int_0^\infty f(t)e^{-zt}dt = 0 \qquad (z \in H_\alpha) \tag{11}$$

then for $z = \alpha + is$ $(s \in R)$ we have

$$\int_0^\infty f(t)e^{-\alpha t}e^{-ist}dt = 0 , \tag{12}$$

By Lemma 1.1.3, $\omega(t) \geq e^{-\alpha t}$. Thus $f(t)e^{-\alpha t} \in L^1(R^+) \subset L^1(R)$ and (12) says that the Fourier transform of $f(t)e^{-\alpha t}$ is 0. Thus, $f(t)e^{-\alpha t} = 0$ or $f = 0$.

Now, we return to the case $\alpha = \infty$, we note that this is equivalent to $\lim\limits_{t\to\infty} \omega(t)^{1/t} = 0$.

1.1.8 Lemma. If $\alpha = \infty$, then $L^1(\omega)$ is a radical algebra.

Proof. [G.R. Allan, cf.9].

Let χ be the characteristic function of $[a, b]$, where $0 < a < b$. If χ^{*n} denotes the n times product of χ under the convolution product then a simple induction shows that the support of χ^{*n} is equal to $[na, nb]$, and that $|\chi^{*n}(t)| \leq (b - a)^{n-1}$ for $t \in [na, nb]$, so that

$$\|\chi^{*(n)}\| \leq (b - a)^{n-1} \int_{na}^{nb} \omega(t)dt$$

Given $e^{-1} > \varepsilon > 0$, choose t_0 so that $\omega(t)^{1/t} < \varepsilon$ $(t \geq t_0)$.
If $na > t_0$, $\|\chi^{*n}\| \leq (b - a)^{n-1} \int_{na}^{nb} \varepsilon^t dt \leq (b - a)^{n-1} \dfrac{\varepsilon^{na}}{|\log\varepsilon|}$,
so that $\lim \|\chi^{*n}\|^{1/n} \leq (b - a)\varepsilon^a$. But this is true for each $\varepsilon > 0$, so that χ is quasinilpotent. Since linear combinations of such functions are dense in $L^1(\omega)$, and $L^1(\omega)$ is commutative the result follows.

In the rest of this chapter unless otherwise stated e_n will be the functions $n\chi_{[0,\ 1/n]}$ $(n = 1, 2, 3, \ldots)$.

1.1.9 Lemma. $\{e_n : n \in \underset{\sim}{N}\}$ is a bounded approximate identity for $L^1(\omega)$.

Proof. We have to show that for every $f \in L^1(\omega)$,

$$||f - e_n * f|| \to 0 \quad \text{as} \quad n \to \infty$$

Since $C_c(R^+)$ [the space of continuous functions on R^+ with compact support] is dense in $L^1(\omega)$, we can assume $f \in C_c(R^+)$. Then

(1) $$(f*e_n)(x) = n\int_0^x e_n(y) f(x-y)\,dy = \begin{cases} n\int_0^x f(x-y)\,dy & x \le \frac{1}{n} \\ n\int_0^{\frac{1}{n}} f(x-y)\,dy & \frac{1}{n} \le x \end{cases}$$

(2) $$\begin{aligned} ||f - f*e_n|| &= \int_0^\infty |f(x) - (f*e_n)(x)|\,\omega(x)\,dx \\ &= \int_0^{\frac{1}{n}} |f(x) - n\int_0^x f(x-y)\,dy|\,\omega(x)\,dx \\ &\quad + \int_{\frac{1}{n}}^\infty |f(x) - n\int_0^{\frac{1}{n}} f(x-y)\,dy|\,\omega(x)\,dx \\ &\le \int_0^{\frac{1}{n}} |f(x)|\,\omega(x)\,dx + n\int_0^{\frac{1}{n}}\int_0^x |f(x-y)|\,dy\ \omega(x)\,dx + \\ &\quad + n\int_{\frac{1}{n}}^\infty \int_0^{\frac{1}{n}} |f(x) - f(x-y)|\,dy\ \omega(x)\ dx \ . \end{aligned}$$

The first integral in the above sum tends to 0 , as the interval of integration tends to 0 . The second integral by the boundedness of f and ω , and the third integral by the uniform continuity of f tends to 0 .

In the next proposition (Proposition 1.1.12) we prove that if θ is an endomorphism of $L^1(\omega)$, then θ has an extension to an endomorphism $\overline{\theta}$ of $M(\omega)$. We will use this fact to study the endomorphisms of $L^1(R^+)$ [Theorem 1.2.10] and to characterize the isometric isomorphisms of $L^1(\omega)$ when $L^1(\omega)$ is a radical algebra [theorem 1.3.11], first we need the following two lemmas.

Lemma 1.1.10 The product in $M(\omega)$ is separately σ-continuous, i.e. if $\{\mu_\lambda : \lambda \in \Lambda\}$ is a net in $M(\omega)$ and $\mu_\lambda \overset{\sigma}{\to} \mu$, then for every $\nu \in M(\omega)$, $\mu_\lambda * \nu \overset{\sigma}{\to} \mu * \nu$.

Proof. For every $\psi \in C_0(\omega)$, we have

$$(1) \qquad \int_{R^+} \psi(x)\ d\mu_\alpha(x) \to \int_{R^+} \psi(x)\ d\mu(x) .$$

Now if $\phi \in C_0(\omega)$ we have to show that

$$(2) \qquad \int_{R^+} \phi(x)\ d(\nu*\mu_\alpha)(x) \to \int_{R^+} \phi(x)\ d(\nu*\mu)(x) ,$$

or equally we have to show that

$$(3) \qquad \int_{R^+}\int_{R^+} \phi(x+y)\ d\nu(x)\ d\mu_\alpha(y) \to \int_{R^+}\int_{R^+} \phi(x+y)\ d\nu(x)\ d\mu(y) .$$

To prove (3) we show that the function ψ defined by

$$\psi(y) = \int_{R^+} \phi(x+y)\ d\nu(x) ,$$

is in $C_0(\omega)$. Since ν is a linear combination of four positive measures each of which is in $M(\omega)$ without loss of generality we can assume that ν is positive. Then

$$(4)\quad \int_{R^+} \phi(x+y)\, d\nu(x) = \int_{R^+} \frac{\phi(x+y)}{\omega(x+y)}\, \omega(x+y)\, d\nu(x)$$

$$\leq \omega(y) \int_{R^+} \frac{\phi(x+y)}{\omega(x+y)}\, \omega(x)\, d\nu(x)$$

$$\leq \omega(y)\, \|\phi\| \int_{R^+} \omega(x)\, d\nu(x) \cdot$$

Now, by the Lebesque's dominated convergence theorem we have

$$(5)\quad \lim_{y\to y_0} \int_{R^+} \phi(x+y)\, d\nu(x) = \int_{R^+} \lim_{y\to y_0} \phi(x+y)\, d\nu(x)$$

$$= \int_{R^+} \phi(x+y_0)\, d\nu(x)$$

and

$$\lim_{y\to\infty} \int_{R^+} \phi(x+y)\, d\nu(x) = \int_{R^+} \lim_{y\to\infty} \phi(x+y)\, d\nu(x) = 0\ .$$

1.1.11 Lemma. $L^1(\omega)$ isbso dense in $M(\omega)$.

Proof. For every $\mu \in M(\omega)$, $\mu * e_n \in L^1(\omega)$. If $f \in L^1(\omega)$ then

$$\|\mu * f - (\mu * e_n) * f\| = \|\mu * (f - e_n * f)\| \leq \|\mu\|\ \|f - e_n * f\| \to 0$$

1.1.12 Proposition. Let θ be a continuous endomorphism of $L^1(\omega)$. Then

(I) θ has an extension to a continuous endomorphism $\overline{\theta}$ of $M(\omega)$.

(II) $\overline{\theta}$ is continuous from $[M(\omega)\ ;\ \text{bso}]$ into $[M(\omega)\ :\ \sigma]$.

Proof. We prove this proposition in two steps.

First step. We prove that $\theta(e_n) \overset{\sigma}{\to} \lambda$, where λ is a measure in $M(\omega)$ with $\lambda^2 = \lambda$ and $\lambda * \theta(f) = \theta(f)$ for every $f \in L^1(\omega)$.

Since $\{\theta(e_n): n=1, 2, \dots\}$ is bounded, by the σ-compactness of the unit ball of $M(\omega)$, it has a σ-limit point λ. Thus, there is a subsequence $\{\theta(e_{n_k}) : k = 1, \dots\}$ such that $\theta(e_{n_k}) \xrightarrow{\sigma} \lambda$ [since $C_0(\omega)$ is separable the unit ball of $M(\omega)$ is metrizable and this guarantees the existence of the subsequence]. For every $f \in L^1(\omega)$ by lemma 1.1.10 we have

$$\theta(e_{n_k}) * \theta(f) \to \lambda*\theta(f)$$

(1)

$$\theta(e_{n_k}) * \theta(f) = \theta(e_{n_k} * f) \xrightarrow{||\cdot||} \theta(f).$$

Thus,

(2) $$\lambda * \theta(f) = \theta(f) \qquad (f \in L^1(\omega))$$

In particular for $f = e_{n_k}$ we have (3) $\lambda * \theta(e_{n_k}) = \theta(e_{n_k})$.

If we compute the σ-limit of both sides by lemma 1.1.10 we obtain $\lambda^2 = \lambda$. If η is another σ-limit point then an argument as above shows that $\eta*\lambda = \lambda*\eta = \eta = \lambda$. Thus λ is the only σ-limit point of $\theta(e_n)$, and $\theta(e_n) \xrightarrow{\sigma} \lambda$.

Second step. For each $\mu \in M(\omega)$ the limit $\lim_\sigma \theta(\mu*e_n)$ exists and $\overline{\theta}$ defined by $\overline{\theta}(\mu) = \lim_\sigma \theta(\mu*e_n)$, $(\mu \in M(\omega))$ satisfies (I) and (II).

Proof. Since $\{\theta(\mu*e_n) : n = 1, 2, \dots\}$ is bounded it has a σ-limit point λ_μ and there is a subsequence $\{e_{n_k} : k = 1, 2, \dots\}$ such that

$$\theta(\mu*e_{n_k}) \xrightarrow{\sigma} \lambda_\mu$$

Then for every $f \in L^1(\omega)$ by lemma 1.1.10 we have

$$\theta(\mu * e_{n_k}) * \theta(f) \overset{\sigma}{\to} \lambda_\mu * \theta(f) .$$

But, $$\theta(\mu * e_{n_k}) * \theta(f) = \theta(\mu * e_{n_k} * f) \overset{\|\cdot\|}{\to} \theta(\mu * f) .$$

Therefore, $\theta(f) * \lambda_\mu = \theta(\mu * f)$.

In particular, for $f = e_n$, we obtain,

$$\theta(e_n) * \lambda_\mu = \theta(\mu * e_n) .$$

From here by letting $n \to \infty$ and by first step we obtain

$$\lim_{\sigma} \theta(\mu * e_n) = \lambda * \lambda_\mu = \lambda_\mu .$$

It is easy to verify that $\overline{\theta}$ is an extension of θ and is an endomorphism of $M(\omega)$. So far we have proved (I). To prove (II) observe that if $\{f_\alpha : f_\alpha \in L^1(\omega), \alpha \in A\}$ is a net and $f_\alpha \overset{bso}{\to} \mu \in M(\omega)$, then $\overline{\theta}(f_\alpha) \overset{\sigma}{\to} \overline{\theta}(\mu)$. Thus if W is an open σ-neighbourhood of zero there is an open bso-neighbourhood of zero V such that

(1) $\overline{\theta}\,[(\mu + V) \cap L^1(\omega)] \subset \overline{\theta}(\mu) + W$

Now, let $W' \subset W$ be an open σ-neighbourhood of zero such that $W' = -W'$ and $W' + W' \subset W$, and let U be an open bso-neighbourhood of zero in $M(\omega)$ such that

(2) $\overline{\theta}\,[(\mu + U) \cap L^1(\omega)] \subset \overline{\theta}\mu + W'$

If $\lambda \in \mu + U$, we can find a bso-neighbourhood U_λ of zero such that

(3) $\overline{\theta}((\lambda + U_\lambda) \cap L^1(\omega)) \subset \overline{\theta}\lambda + W'$

and $\lambda + U_\lambda \subset \mu + U$.

Then, we have

$$(4)\qquad \overline{\theta}((\lambda + U_\lambda) \cap L^1(\omega)) \subset \overline{\theta}((\mu + U) \cap L^1(\omega)) \subset \overline{\theta}\mu + W'$$

and $\overline{\theta}((\lambda + U_\lambda) \cap L^1(\omega)) \subset \overline{\theta}\lambda + W'$, which together with lemma 1.1.11 imply that

$$(5)\qquad (\overline{\theta}\lambda + W') \cap (\overline{\theta}\mu + W') \neq \emptyset$$

which means that $\overline{\theta}\lambda \in \overline{\theta}\mu + W$. Hence, $\overline{\theta}(\mu + U) \subset \overline{\theta}\mu + W$ and this proves II.

Note 1.1.13. The result of proposition 1.1.12 can be stated in a more general form. If A is a commutative Banach algebra then a continuous linear operator T on A is said to be a multiplier if for every $x, y \in A$,

$$T(x.y) = x.T(y) = T(x).y .$$

The space of all multipliers on A is subalgebra of $B(A)$ [bounded linear operators on A] called the multiplier algebra of A and denoted by $M(A)$, for each $x \in A$, let the operator T_x be defined by

$$T_x(y) = x.y \qquad (y \in A)$$

Then T_x is a multiplier on A. If A has a bounded approximate identity bounded by 1 , then the map $x \to T_x$ $(x \in A)$ is an isometric embedding of A in $M(A)$. This is the case that we will be concerned with. For example, $g_n = \dfrac{e_n}{\|e_n\|}$ $(n \in \mathbb{N})$ is a bounded approximate identity of norm 1 for $L^1(\omega)$. Moreover in this case $M(\omega)$ is the multiplier algebra of $L^1(\omega)$. For if μ is a measure in $M(\omega)$, then the map

$$T_\mu(f) = f*\mu \qquad (f \in L^1(\omega))$$

is a multiplier on $L^1(\omega)$ moreover $||T_\mu|| = ||\mu||$. On the other hand if T is a multiplier on $L^1(\omega)$, then the method used in the proof of proposition 1.1.12 shows that $T(g_n)$ tends to a measure $\mu_T \in M(\omega)$ in the topology σ, moreover $T(f) = \mu * f$ $(f \in L^1(\omega))$.

1.1.12′ Proposition. Let A be a commutative Banach algebra with a bounded approximate identity bounded by 1 and let the multiplier algebra $M(A)$ of A be the dual of a Banach space X, if multiplication in $M(A)$ is separately continuous in the topology $\sigma = \sigma[M(A), X]$, then every endomorphism θ of A has an extension to an endomorphism $\bar{\theta}$ of $M(A)$.

Proof. Similar to the proof of 1.1.12.

CHAPTER 1.2

Semi-simple weighted algebras

In this chapter we will study homomorphisms from one semi-simple algebra $L^1(\omega_1)$ into another semi-simple algebra $L^1(\omega_2)$ and at the end we will specialize to the endomorphisms of $L^1(\omega)$ with $w(t) = 1 (t \in R^+)$.

1.2.1 Theorem. If $L^1(\omega_1)$ and $L^1(\omega_2)$ are two semi-simple weighted algebras with $\alpha_i = \lim_{t\to\infty} - \frac{1}{t} \log \omega_i(t)$ $(i = 1, 2)$, then for every non-zero isomorphism θ of $L^1(\omega_1)$ onto $L^1(\omega_2)$ there exist $A > 0, B \geq 0$, such that

$$(1) \quad (\theta f)(t) = \frac{1}{A} f(\frac{t}{A}) e^{-[\frac{1}{A}(iB + \alpha_1) - \alpha_2]t} \qquad (f \in L^1(\omega_1), t \in R^+)$$

Proof. For every $z \in H_{\alpha_2}$, the mapping

$$(2) \qquad f \to \int_0^\infty (\theta f)(t) e^{-zt} dt \qquad (f \in L^1(\omega_1))$$

defines a multiplicative linear functional on $L^1(\omega_1)$, which is not identically zero since θ is an isomorphism. Thus, there is $\tilde{\theta}(z) \in H_{\alpha_1}$, such that,

$$(3) \qquad \int_0^\infty (\theta f)(t) e^{-zt} dt = \int_0^\infty f(t) e^{-\tilde{\theta}(z)t} dt$$

By lemma 1.1.3, there is a number $\beta > 0$ such that the function f defined by $f(t) = e^{-\beta t}$ $(t \in R^+)$ is in $L^1(\omega_1)$. For the function f (3) becomes,

$$(4) \qquad \int_0^\infty (\theta f)(t) e^{-zt} = \frac{1}{\beta + \tilde{\theta}(z)}$$

The left hand side of (4) defines an analytic function in the interior of H_{α_2}, therefore $\tilde{\theta}(z)$ is analytic in the interior of H_{α_2} and since θ is one-to-one and onto $\tilde{\theta}$ is one-to-one and onto. The two half planes H_{α_1} and H_{α_2} are conformally equivalent to the unit disc and the conformal mappings of the unit disc are known to be the maps $\omega_{a,b}(z) = \dfrac{az + b}{\bar{b}z + \bar{a}}$ with $|a|^2 - |b|^2 = 1$, [cf.23, Th. 7.20, p.186]. Thus we can compute all the conformal mappings from H_{α_2} onto H_{α_1} and they are given by

$$(5) \qquad \tilde{\theta}(z) = \frac{a(z - \alpha_2) + ib}{ci(z - \alpha_2) + d} + \alpha_1 \qquad (a,\ b,\ c,\ d \in R,\ ad + bc \geq 0)$$

The number c cannot be any number. Indeed $c = 0$, for if $c \neq 0$ we let $z = \alpha_2 + is$ $(s \in R)$ in (3) then

$$(6) \qquad \int_0^\infty (\theta f)(t) e^{-\alpha_2 t - ist} dt = \int_0^\infty f(t) e^{-\left(\frac{isa + ib}{-cs + d} + \alpha_1\right)t} dt$$

By lemma 1.1.3, $e^{-\alpha_2 t} \leq \omega_2(t)$, thus $(\theta f)(t)e^{-\alpha_2 t} \in L^1(R^+) \subset L^1(R)$ and the right hand side of (6) can be regarded as the Fourier transform of a function in $L^1(R)$. Thus, when $s \to \infty$ by Riemann Lebesgue lemma we obtain

$$(7) \qquad 0 = \int_0^\infty f(t)\ e^{\frac{ai}{c}t}\ e^{-\alpha_1 t} dt \qquad (f \in L^1(\omega))$$

If we interchange f with $|f|\ e^{-\frac{ai}{c}t}$ we obtain

$$(8) \qquad \int_0^\infty |f(t)| e^{-\alpha_1 t} dt = 0 \qquad (f \in L^1(\omega_1))$$

Thus, $f = 0$. From this contradiction we obtain $c = 0$ and (6) becomes

(9) $$\int_0^\infty (\theta f)(t)e^{-\alpha_2 t} e^{-ist}dt = \int_0^\infty f(t)e^{-(is\frac{a}{d} + i\frac{b}{d} + \alpha_1)t} dt$$

Now, let $A = \frac{a}{d}$, $B = \frac{b}{d}$, and a change of variable $At = u$ in the right hand side of (9) gives,

(10) $$\int_0^\infty (\theta f)(t)e^{-\alpha_2 t} e^{-ist}dt = \frac{1}{A}\int_0^\infty f(\tfrac{u}{A})e^{-\frac{1}{A}(iB + \alpha_1)u} e^{-isu}du$$

Now, $(\theta f)(t)e^{-\alpha_2 t}$ as well as $\frac{1}{A} f(\frac{t}{A}) e^{-\frac{1}{A}(iB + \alpha_1)t}$ are in $L^1(R)$, thus the semi-simplicity of $L^1(R)$ implies

$$(\theta f)(t) = \frac{1}{A} f(\tfrac{t}{A}) e^{-[\frac{1}{A}(iB + \alpha_1) - \alpha_2]t}$$

as asserted.

1.1.2 The algebra $L^1(R^+)$. In the particular case $\omega(t) = 1$ ($t \in R^+$), we use the standard notation $L^1(R^+)$ for the algebra $L^1(\omega)$. This algebra can be regarded as a closed subalgebra of the group algebra $L^1(R)$. Indeed for $f \in L(R^+)$ let $\tilde{f}$ be defined by

(1) $$\tilde{f}(x) = \begin{cases} f(x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

then, the map $f \to \tilde{f}$ is an isometric embedding of $L^1(R^+)$ into $L^1(R)$. The algebra $L^1(R^+)$ has been studied by several authors, Newman, Schwartz and Shapiro have studied its topological generators [cf.26], Wermer [cf.39] and Simon [cf.32] independently have proved its maximality in $L^1(R)$ and Sinclair has shown that if A is a separable Banach algebra then A has a bounded approximate identity bounded by 1 if and only if there is a homomorphism θ from $L^1(R^+)$ into A such that $\theta(L^1(R^+)).A = A.\ \theta(L^1(R^+))$ and $\|\theta\| = 1$ [cf.33].

The problem of characterizing all the generators of $L^1(R^+)$ as far as we know is still unsolved. However, we need only know the following result about a topological generator of $L^1(R^+)$.

1.2.3 Theorem (Rudin, cf.27, Th.9, 2.3 p.234). The function f defined by $f(t) = e^{-t}$ $(t \in R^+)$ is a topological generator of the algebra $L^1(R^+)$.

Proof. Let $\alpha(x) = 2f(x)$ if $x \geq 0$ and $\alpha(x) = 0$ if $x < 0$, and put $\beta(x) = \alpha(-x)$ $(x \in R)$. Then $\hat{\alpha}(y) = 2(1 + iy)^{-1}$ $\hat{\beta}(y) = 2(1 - iy)^{-1}$, and so

$$(1) \qquad \alpha + \beta = \alpha * \beta$$

The derivatives of $\hat{\alpha}$ are constant multiples of powers of $\hat{\alpha}$. Hence, writing $\alpha^1 = 1$, $\alpha^n = \alpha^{n-1} * \alpha$ $(n = 2, 3, \ldots)$ we have

$$(2) \qquad \alpha^n(x) = c_n x^{n-1} \alpha(x) \qquad (n = 1, 2, 3, \ldots)$$

the constants c_n being different from 0. Suppose $\phi \in L^\infty(R)$, $\phi(x) = 0$ for $x < 0$, and $\int_0^\infty \alpha^n(x)\phi(x)dx = 0$ for $n = 1, 2, 3, \ldots$. The function

$$F(z) = \int_0^\infty e^{-xz}\phi(x)dx$$

is then analytic in the right-half-plane, and since

$$(3) \qquad F^{(n)}(1) = \frac{(-1)^n}{2c_n} \int_0^\infty \alpha^{n+1}(x)\phi(x)dx = 0 \qquad (n = 0, 1, 2, \ldots)$$

F is identically 0. In particular, this is so for $F(1 + iy)$, the Fourier Transform of $e^{-x}\phi(x)$. Hence $\phi = 0$, and we can conclude that α is a topological generator of $L^1(R^+)$. Thus the function f is a topological generator of $L^1(R^+)$.

Next, we prove that every non-zero endomorphism of the algebra $L^1(R^+)$ is a monomorphism, first we need the following lemma.

1.2.4 Lemma. Let θ be a non-zero endomorphism of $L^1(R^+)$, then

$$\lim_{\sigma} \theta(e_n) = \delta_0 .$$

Proof. In the proof of proposition 1.1.12 we saw that $\lim_{\sigma} \theta(e_n) = \lambda$ where $\lambda \in M(R^+)$ is an idempotent measure. But, since $M(R^+)$ is a subalgebra of $M(R)$, λ is an idempotent measure in $M(R)$ and the only idempotent measures in $M(R)$ are δ_0 and 0 [cf. 27, Note 3.2.1 p.61]. If $\lim_{\sigma} \theta(e_n) = 0$, then the separate σ-continuity of product in $M(R^+)$ [lemma 1.1.10] implies that $\theta(f) = 0$ for every $f \in L^1(R^+)$. Thus $\lambda \neq 0$, and we conclude $\lambda = \delta_0$.

1.2.5 Theorem. Every non-zero endomorphism of $L^1(R^+)$ is a monomorphism.

Proof. Let $z \in H_0$, with $\mathrm{Re}\, z > 0$. The map

(1) $$f \to \int_0^\infty (\theta f)(t)e^{-zt}dt \qquad (f \in L^1(R^+))$$

is a multiplicative linear functional on $L^1(R^+)$. Moreover, it is not identically zero, since otherwise we have

(2) $$\int_0^\infty (\theta f)(t)e^{-zt}dt = 0 \qquad (f \in L^1(R^+))$$

and for $f = e_n$ (2) becomes,

$$(3) \qquad \int_0^\infty (\theta(e_n))(t)e^{-zt}dt = 0 \qquad (n = 1, 2, 3, \ldots)$$

Since $\mathrm{Re}z > 0$, $e^{-zt} \in C_0(R^+)$ and by Lemma 1.2.4 we get

$$(4) \qquad 0 = \int_0^\infty (\theta(e_n))(t)e^{-zt}dt \to \int_0^\infty e^{-zt}d\delta_0(t) = 1$$

which is a contradiction. Thus for every z, with $\mathrm{Re}z > 0$ the map (1) defines a character on $L^1(R^+)$, therefore there is a number say $\tilde{\theta}(z) \in H_0$ such that

$$(5) \qquad \int_0^\infty (\theta f)(t)e^{-zt}dt = \int_0^\infty f(t)e^{-\tilde{\theta}(z)t}dt \cdot \qquad (f \in L^1(R^+))$$

The map $z \to \tilde{\theta}(z)$ defines an analytic function in the $\mathrm{Int}H_0$ (the interior of H_0) [see the proof of theorem 1.2.1]. Thus $\tilde{\theta}(\mathrm{Int}\ H_0)$ is either an open subset of H_0 or a single point in H_0. If $\tilde{\theta}(\mathrm{Int}\ H_0)$ was a single point, by letting $z \to \infty$ in both sides of (5) we get

$$(6) \qquad \int_0^\infty (\theta f)(t)e^{-zt}dt = 0 \qquad (z \in \mathrm{Int}\ (H_0))$$

Also the continuity of $z \to \int_0^\infty (\theta f)(r)e^{-zt}dt$ implies

$$(7) \qquad \int_0^\infty (\theta f)(t)e^{-zt}dt = 0 \qquad (z \in H_0)$$

From semi-simplicity of $L^1(R^+)$ and (7) we get $\theta f = 0$ $(f \in L^1(R^+))$ which is a contradiction. Thus, $\tilde{\theta}(\mathrm{int}\ (H_0))$ is an open subset of $\mathrm{Int}(H_0)$. To prove that $\theta(f)$ is a monomorphism let $\theta(f) = 0$, and consider the function

$$F(z) = \int_0^\infty f(t)e^{-zt}dt \qquad (z \in H_0)$$

then F is analytic in $\text{Int}\,(H_0)$, and continuous on H_0. On the other hand F is zero in $\tilde{\theta}(\text{Int}\,H_0)$ which is an open subset of $\text{Int}\,H_0$ thus by the uniqueness theorem for analytic functions $F(z) \equiv 0 \ (z \in H_0)$. Now semi-simplicity of $L^1(R^+)$ implies $f = 0$.

The fact that for every endomorphism θ of $L^1(R^+)$ and $z \in \text{Int}\,H_0$, the mapping

$$f \to \int_0^\infty \theta f(t)e^{-zt}dt \qquad (f \in L\ (R^+)),$$

is a character on $L^1(R^+)$ is useful in finding a general formula for the endomorphisms of $L^1(R^+)$. In this case the formula (5) of 1.2.5 is valid and this defines an analytic map $\tilde{\theta}$ from $\text{Int}(H_0)$ into $\text{Int}(H_0)$. Formula (5) says that $\tilde{\theta}$ is such that for every $f \in L^1(\omega)$ the function $F(s) = \int_0^\infty f(t)e^{-\tilde{\theta}(s)t}dt\,(s>0)$ is the Laplace transform of a function in $L^1(R^+)$. There are necessary and sufficient conditions under which a function defined on R^+ is the Laplace transform of a function in $L^1(R^+)$ and we can translate these to necessary and sufficient conditions on $\tilde{\theta}$ such that the function F defined as above becomes the Laplace transform of a function in $L^1(R^+)$. Conversely, let $\tilde{\theta}(z)$ be an analytic map from $\text{Int}(H_0)$ into $\text{int}\,H_0$ such that for every $f \in L^1(\omega)$, $F(s) = \int_0^\infty f(t)e^{-\tilde{\theta}(s)t}dt\ (s>0)$ is the Laplace transform of a function in $L^1(R^+)$, and define $\theta(f)$ to be the inverse Laplace transform of $F(s) = \int_0^\infty f(t)e^{-\tilde{\theta}(s)t}dt$ then θ is an endomorphism of $L^1(R^+)$. The above discussion leads to a characterization (although, not very nice!) of the endomorphisms of $L^1(R^+)$. First we need a definition and a necessary and sufficient condition for a function $f(s)$ $(s>0)$ to be the Laplace transform of a function in $L^1(R^+)$ and an inversion formula, [cf. 40].

Definition 1.2.6 Let f be an infinitely differentiable function on $[0, \infty)$, for every positive number t and every positive integer k, we define the operator $L_{k,t}$ by

$$L_{k,t}[f] = (-1)^k \left(\frac{k}{t}\right)^{k+1} f^{(k)} \left(\frac{k}{t}\right)$$

[$f^{(n)}(x)$ denotes the nth derivative of f at x].

Definition 1.2.7 A function f defined on $[0, \infty)$ satisfies conditions W if it is infinitely differentiable in $(0, \infty)$ vanishes at infinity, and if

$$\int_0^\infty |L_{k,t}[f]| \, dt < \infty \qquad (k = 1, 2, 3, \ldots)$$

$$\lim_{\substack{j\to\infty \\ k\to\infty}} \int_0^\infty |L_{k,t}[f] - L_{j,t}[f]| \, dt = 0$$

Theorem 1.2.8 (Widder) Conditions W are necessary and sufficient that

$$f(x) = \int_0^\infty e^{-xt} \phi(t) \, dt ,$$

where $$\int_0^\infty |\phi(t)| \, dt < \infty$$

Proof. [cf. 40 Th.17a p.318].

Theorem 1.2.9 (Inversion formula). Under the hypothesis of Theorem 1.2.8

$$\phi(y) = \lim_{k\to\infty} \int_0^\infty e^{-xy} P_{2k-1}(xy) f(x)\,dx$$

for almost all positive y, where $P_{2k-1}(t)$ is defined by

$$P_{2k-1}(t) = \frac{(-1)^{k-1}(2k-1)!}{k!\ (k-2)!} \sum_{p=0}^{k} \binom{k}{p} \frac{(-t)^{2k-p-1}}{(2k-p-1)!}$$

Proof. [cf. 40 Th. 25b p. 386]

If we combine the equation (5) of theorem 1.2.4 and theorems 1.2.8 and 1.2.9 we obtain.

1.2.10 Theorem. Every endomorphism θ of $L^1(R^+)$ is given by

$$(\theta f)(x) = \lim_{k\to\infty} \int_0^\infty e^{-xy} P_{2k-1}(xy) \int_0^\infty f(t) e^{-\tilde{\theta}(y)t} dt\ dy$$

$$(f \in L^1(R^+),\ x \in R^+)$$

where $\tilde{\theta}$ is an analytic function from $(\mathrm{Int}\ H_0)$ into $\mathrm{Int}(H_0)$ which satisfies the following conditions,

$$\lim_{s\to\infty} \int_0^\infty f(t) e^{-\tilde{\theta}(s)t} dt = 0 \qquad (f \in L^1(R^+))$$

$$\int_0^\infty \left| L_{k,t} \left(\int_0^\infty f(x) e^{-\tilde{\theta}(t)x} \right) \right| dt < \infty \qquad (f \in L^1(R^+),\ k \in N)$$

and

$$\lim_{k,j\to\infty} \left\| L_{k,t} \int_0^\infty f(x) e^{-\tilde{\theta}(y)x} dx - L_{j,t} \int_0^\infty f(x) e^{-\tilde{\theta}(y)x} dx \right\| = 0$$

$$(f \in L^1(R^+)).$$

There are three types of endomorphisms of $L^1(R^+)$ that we can give explicit formulae for them.

I. Let $t \to g^t : [0, \infty) \to L^1(R^+)$ be a semigroup, if $\{g^t : t \in R^+\}$ is bounded and $t \to g^t$ is measurable the map $f \to \int_0^\infty f(t)g^t dt$ defines an endomorphism in $L^1(R^+)$, where the integral is the integral of a vector valued function. We denote the class of all such endomorphisms by SG. The existence of the semi-groups $t \to g^t$ with the above property is guaranteed by a theorem of Sinclair which is a generalized form of Cohen's factorization theorem and a corollary of which says that in a Banach algebra A with bounded approximate identity for every element $x \in A$, we have a factorization $x = g^t h_t$, $(t \in R^+)$, with $t \to g^t$ $(t \in R^+)$ a bounded continuous semigroup. For a more general form of Sinclair's theorem see [cf. 33].

A concrete example of a semi-group in $L^1(R^+)$ is g^t defined by

$$g^t(x) = \frac{1}{2}\pi^{-\frac{1}{2}} t\, x^{-\frac{3}{2}} e^{-\frac{t^2}{4x}} \qquad (t \in R^+,\ x \in R^+)$$

This is a semi-group because the Laplace transform of g^t is $(Lg^t)(p) = e^{-tp^{\frac{1}{2}}}$. Thus, $(Lg^{t+s})(p) = (Lg^t)(p)(Lg^s)(p)$.

In this example the map $t \to g^t$ from R^+ into $L^1(R^+)$ is continuous and for each $t \in R^+$ we have $||g^t|| = 1$.

II. Let ψ be a continuous semi-character on R^+ (a continuous bounded semi-group homomorphism of R^+ into $\underset{\sim}{C}$). Then there is a $z \in H_0$ such that $\psi(t) = e^{-zt}$ $(t \in R^+)$. Now let θ_z be defined by,

(1) $$(\theta_z f)(x) = e^{-zx} f(x) \qquad (f \in L^1(R^+),\ x \in R^+)$$

Then it is easy to verify that θ defines an endomorphism of $L^1(R^+)$. We denote the class of all such θ_z by SC .

III. Finally, we introduce the class H , which arises from the continuous semi-group homomorphisms of R^+. For every $\alpha \in R^+$, we define θ_α by

$$(\theta_\alpha f)(x) = \alpha f(\alpha x) \qquad (f \in L^1(R^+),\ x \in R^+)$$

Then θ_α is an endomorphism of $L^1(R^+)$.

The intersection of each of the above classes with the other two is either empty or the identity endomorphism. To see what is $SG \cap SC$, let $\theta \in SG \cap SC$, then there is a $z \in H_0$ and a semi-group g^t in $L^1(R^+)$ such that,

(1) $$\theta(\ f)(x) = e^{-zx} f(x) \qquad (f \in L^1(\omega),\ x \in R^+)$$

(2) $$\theta f = \int_0^\infty f(t) g^t dt \qquad (f \in L^1(\omega))$$

For every $s \in R^+$, let χ_s be the character on $L^1(R^+)$, which is given by

$$\chi_s(f) = \int_0^\infty f(x) e^{-isx} dx \qquad (f \in L^1(R^+))$$

If we apply χ_s to both sides of (1) and (2) we obtain,

(3) $$\int_0^\infty f(t) e^{-(z+is)t} dt = \int_0^\infty f(t) \chi_s(g^t) dt = \int_0^\infty f(t) \int_0^\infty g^t(x) e^{-isx} dx\, dt ,$$

Thus,

(4) $$e^{-(z+is)t} = \int_0^\infty g^t(x) e^{-isx} dx .$$

Now, by the Riemann-Lebesgue lemma the right hand side of (4) tends to 0 as $s \to \infty$, while the left-hand side oscillates, so that $SG \cap SC = \emptyset$.

If $\theta \in SG \cap H$, then a similar argument to above shows that for $s \in R$,

$$e^{-\frac{ist}{\alpha}} = \int_0^\infty g^t(x) e^{-isx} dx .$$

Again an application of the Riemann-Lebesgue lemma shows that this equality is impossible.

Finally, let $\theta \in SC \cap H$, then there is a $z \in H_0$, $\alpha \in R^+$, such that

$$(\theta f)(x) = \alpha f(\alpha x) \quad \text{and} \quad (\theta f)(x) = e^{-zx} f(x) \qquad (f \in L^1(R^+),\ x \in R^+)$$

Then

$$\alpha f(\alpha x) = e^{-zx} f(x) \qquad (f \in L^1(R^+),\ x \in R^+)$$

In particular for the function f defined by $f(t) = e^{-t}$ $(t \in R^+)$ we must have

$$\alpha e^{-\alpha x} = e^{-(z+1)x} \qquad (x \in R^+)$$

If $x \to 0$ in both sides of this equation we get $\alpha = 1$ and this implies $z = 0$. Therefore, the identity is the only endomorphism of $L^1(R^+)$ which is in $SC \cap H$.

Perhaps the classes SG, SC, H generate the algebra of all endomorphisms. Even if so the closure of polynomials in elements of SG, SC, H, seems to us not easier to express than what we have in theorem 1.2.10.

We also note that if $\mathrm{re}\,z > 0$, then multiplication by e^{-zt} is an endomorphism of $L^1(R^+)$ which is not an automorphism, and this gives a large class of endomorphisms which are not automorphisms.

CHAPTER 1.3

Isometric isomorphisms of radical $L^1(\omega)$

In Chapter 1.2 we characterized all of the isomorphisms from $L^1(\omega_1)$ onto $L^1(\omega_2)$, when $L^1(\omega_1)$ and $L^1(\omega_2)$ were semi-simple. This, in particular gives all of the automorphisms of a single algebra $L^1(\omega)$ when it is semi-simple. However, the method used in 1.2 is not applicable to radical algebras. But, by using $\omega(0) = 1$, we can characterize all of the isometric isomorphisms of $L^1(\omega)$. The assumption $\omega(0) = 1$, implies that for large values of n , $\|e_n\|$ is close to 1 , thus

$$g_n = \frac{e_n}{\|e_n\|} \qquad (n \in N) ,$$

is a bounded approximate identity for $L^1(\omega)$ bounded by 1 . This will enable us to extend every isometric isomorphism of $L^1(\omega)$ to an isometric isomorphism of $M(\omega)$, and then by finding the images of the extreme points of the unit ball of $M(\omega)$, we find the extended isometric isomorphism and consequently the original one. We start this programme with:

1.3.1 Lemma. The set of extreme points of the unit ball of $M(\omega)$ is

$$\{\frac{\lambda}{\omega(x)}\,\delta_x : x \in R^+, |\lambda| = 1\}$$

Proof. First we show that if μ is an extreme point of the unit ball of $M(R^+)$, then $\mu = \lambda\delta_x$ $(|\lambda| = 1, x \in R^+)$. Suppose that the support of μ contains two points t_1 and t_2 with $t_1 \neq t_2$. Let U_1 and U_2 be two open sets which separate t_1 and t_2 and let

$$m = |\mu|(U_1) \;,\quad \alpha = \frac{(\mu|U_1)}{m} \;,\quad \beta = \frac{\mu - m\alpha}{1 - m}$$

Then $0 < m < 1$, $||\alpha|| \le 1$, $||\beta|| \le 1$, and $\mu = m\alpha + (1 - m)\beta$. Thus the support of μ reduces to a single point and $\mu = \lambda\delta_x$ with $|\lambda| \le 1$, $x \in R^+$. It is easy to verify that $|\lambda|$ is not less than 1 and every $\lambda\delta_x$ with $|\lambda| = 1$ and $x \in R^+$ is an extreme point. Thus, the set of extreme points of the unit ball of $M(R^+)$ is $\{\lambda\delta_x : |\lambda| = 1, x \in R^+\}$. On the other hand the map

$$d\mu(t) \to \omega(t)\, d\mu(t) : M(\omega) \to M(R^+)$$

is a (linear) isometric isomorphism of $M(\omega)$ onto $M(R^+)$. Thus, if μ is an extreme point of the unit ball of $M(\omega)$, its image under the above map must be an extreme point of the unit ball of $M(R^+)$. Thus, there exists $\lambda \in \underset{\sim}{C}$, $|\lambda| = 1$ and $x \in R^+$, such that

$$\omega(t)\, d\mu(t) = \lambda d\delta_x(t)$$

Therefore, $\mu = \dfrac{\lambda}{\omega(x)}\,\delta_x$ with $|\lambda| = 1$, $x \in R^+$.

Lemma 1.3.2 Let $\mu \in M(\omega)$. If $x \in s(\mu)$ (the support of μ), then for every open neighbourhood U of x, there exists $\psi \in C_0(\omega)$, such that ψ vanishes outside U and $\langle\mu, \psi\rangle \neq 0$.

Proof. There exists a finite interval (a,b) such that $x \in (a,b) \subset U$, [if $x = 0$, then ~~there exists~~ an interval of the form $[0,b)$] and $\mu((a,b)) \neq 0$, by the definition of support. Now for positive δ which is small enough, let ψ be a continuous function defined by

$$\psi(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{\delta} & a < x < a+\delta \\ 1 & a+\delta \le x \le b-\delta \\ \dfrac{b-x}{\delta} & b-\delta < x < b \\ 0 & b \le x \end{cases}$$

For an appropriate choise of δ, $\langle\mu, \psi\rangle$ is near enough to $\mu((a, b))$ and thus is different from 0 .

1.3.3 Lemma. Let K be an interval of the form $[0, a]$ $(a > 0)$ and let $\{\mu_j : j \in J\}$ be a net in $M(\omega)$ such that $\mu_j \overset{s0}{\to} \mu$ with $||\mu_j|| \le M < \infty$ and $s(\mu_j)$, $s(\mu) \subset K$ $(j \in J)$. If $\psi \in C_0(\omega)$ then $\langle\mu_j, \psi\rangle \to \langle\mu, \psi\rangle$.

Proof. Since $\psi(x)$ is uniformly continuous on R^+ and $\omega(x)$ is bounded below on K by a number $\alpha > 0$ and $\omega(0) = 1$, given $0 < \varepsilon < \frac{1}{\alpha}$, there is a $\delta > 0$ such that for $0 \le x < \delta$ and every $y \in R^+$ we have

(1) $$|\psi(x + y) - \psi(y)| < \varepsilon.\alpha \quad \text{and} \quad |\omega(x) - 1| < \varepsilon.\alpha$$

Now, let f be a function in $L^1(\omega)$ defined by,

$$f(x) = \frac{1}{\delta\omega(x)} \quad \chi_{[0,\ \delta]}(x) \ , \qquad (x \in R^+)$$

Then $||f|| = 1$, and if N is an upper bound for $|\psi(t)|$ we have

(2) $$\left|\int_0^\infty \psi(s+t)f(t)dt - \psi(s)\right| = \left|\int_0^\delta \frac{1}{\delta}\psi(s+t)\frac{1}{\omega(t)}dt - \psi(s)\right|$$

$$\le \left|\int_0^\delta \frac{1}{\delta}\psi(s+t)dt - \psi(s)\right| + \left|\int_0^\delta \frac{1}{\delta}\psi(s+t)\frac{dt}{\omega(t)} - \int_0^\delta \frac{1}{\delta}\psi(s+t)dt\right|$$

$$\le \frac{1}{\delta}\left[\int_0^\delta |\psi(s+t) - \psi(s)|dt + \int_0^\delta |\psi(s+t)| \frac{|\omega(t) - 1|}{\omega(t)} dt\right.$$

$$\le \varepsilon\,\alpha + N\frac{\varepsilon\alpha}{1 - \varepsilon\alpha} \qquad (s \in K) .$$

Thus for a suitable choice of ε we can find a function $f \in L^1(\omega)$, with $||f|| = 1$ and

$$(3) \quad \left|\int_0^\infty \psi(s+t)f(t)dt - \psi(s)\right| < \varepsilon.\alpha \le \varepsilon.\omega(s) \qquad (s \in K).$$

Now.

$$(4) \quad |<\mu_j*f, \psi> - <\mu_j, \psi> = \left|\int_0^\infty \left[\int_0^\infty \psi(s+t)f(t)dt - \psi(s)\right]d\mu_j(s)\right|$$

$$\le \varepsilon\left|\int_K \omega(s)\ d\mu_j(s)\right| < M.\varepsilon .$$

Similarly we have

$$(5) \quad |<\mu*f, \psi> - <\mu, \psi>| \le ||\mu||\ \varepsilon.$$

Thus,

$$(6) \quad |<\mu_j, \psi> - <\mu, \psi>| \le |<\mu_j, \psi> - <\mu_j*f, \psi>| +$$

$$|<\mu_j*f, \psi> - <\mu*f, \psi>| + |<\mu*f, \psi> - <\mu, \psi>|$$

$$< (M + ||\mu|| + 1)\varepsilon, \quad \text{for } j \ge j_0,$$

and this proves the lemma.

Lemma 1.3.4 The map $x \to \frac{1}{\omega(x)}\delta_x$ is so-continuous from R^+ with its usual topology into $[E_{R^+} : so]$, this map is a homeomorphism of R^+ onto $[E_{R^+}\ \sigma]$.

Proof. The proof of the so-continuity of the above map is similar to the proof of lemma 1.1.5.

To prove the σ-continuity of the above map let $f \in C_0(\omega)$ then

$$\int_0^\infty f(x)\frac{1}{\omega(y)}d\delta_y(x) = \frac{f(y)}{\omega(y)}$$

and the continuity of f and ω gives the result. To prove that the above map is one to one let

$$\frac{1}{\omega(x)}\delta_x = \frac{1}{\omega(y)}\delta_y \qquad (x, y \in R^+) ,$$

then $x = y$. To prove that the inverse of this map is continuous let

$$\{\frac{1}{\omega(x_\alpha)}\delta_{x_\alpha} : \alpha \in A\}$$

be a net with

$$\frac{1}{\omega(x_\alpha)}\delta_{x_\alpha} \xrightarrow{\sigma} \frac{1}{\omega(x)}\delta_x .$$

Then for the function f defined by $f(x) = \frac{\omega(x)}{1+x}$, which is in $C_0(\omega)$, we have

$$\int_0^\infty f(t)\,d\delta_{x_\alpha}(t) \rightarrow \int_0^\infty f(t)\,d\delta_x(t)$$

or

$$\frac{1}{1 + x_\alpha} \rightarrow \frac{1}{1 + x} .$$

Thus $x_\alpha \rightarrow x$.

Lemma 1.3.5 If $K = [0, a]$, and if

$$TE_K = \{\frac{\lambda}{\omega(x)}\delta_x : |\lambda| = 1, x \in K\} , \qquad \text{then}$$

$$co[TE_K : so] = co[TE_K : \sigma] = \{\mu \in M(\omega) : ||\mu|| \leq 1, s(\mu) \subset K\}$$

where 'co' stands for the closed convex hull.

Proof. By Lemma 1.3.4 E_K is both (σ) and (so) compact. Thus $co[TE_K : so]$ is compact in the (so) topology, as is $co[TE_K : \sigma]$ in the σ-topology, [cf.12 Ex.3 p.511]. We claim that if $\mu \in co[TE_K : so]$, then $||\mu|| \leq 1$, and $s(\mu) \subset K$

proof of claim. Let $\{\mu_\lambda : \lambda \in \Lambda\}$ be a net such that for each $\lambda \in \Lambda$, μ_λ is a convex linear combination of elements of TE_K and $\mu_\lambda \overset{so}{\rightarrow} \mu$. If $x \in s(\mu)$ and $x \notin [0, a]$ we let $2\delta_1 = x - a$ and let $I = (x - \delta_1, x + \delta_1)$. By lemma 1.3.2 there is a function $\psi \in C_0(\omega)$, such that ψ vanishes outside I and $<\mu, \psi> = \int_0^\infty \psi(x)d\mu(x) \neq 0$. Since the map $y \rightarrow \int_0^\infty \psi(x + y)d\mu(x)$ is continuous there is a δ with $\delta_1 > \delta > 0$ such that for $y \in [0, \delta]$ we have

$$(1) \quad \int_0^\infty \psi(x+y)d\mu(x) \neq 0 \quad \text{and} \quad 0 < m < \left|\int_0^\infty \psi(x+y)d\mu(x)\right| < M < \infty$$

let f be a function defined by

$$f(y) = \begin{cases} \dfrac{1}{\int_0^\infty \psi(x+y)d\mu(x)} & y < \delta \\ 0 & \delta \leq y \end{cases}$$

Then we have

$$(2) \quad <\mu_\lambda * f, \psi> = \int_0^\delta f(z) \int_0^a \psi(z+y)d\mu_\lambda(z)dy = 0 \qquad (\lambda \in \Lambda)$$

and

$$(3) \quad <\mu * f \quad \psi> = \int_0^\delta \left[\frac{1}{\int_0^\infty \psi(t+y)d\mu(t)} \int_0^\infty \psi(z+y)d\mu(z)\right] dy = \delta$$

and this contradicts the fact that $\mu_\lambda * f \overset{||\cdot||}{\rightarrow} \mu * f$. Thus $s(\mu) \subset [0, a]$. To show that $||\mu|| \leq 1$, for every $\varepsilon > 0$, $f \in L^1(\omega)$ there is a λ_0 such that $||\mu_\lambda * f - \mu * f|| < \varepsilon$ for $\lambda \geq \lambda_0$. Therefore, $||\mu * f|| \leq ||\mu_\lambda||\ ||f|| + \varepsilon$, $(\lambda \geq \lambda_0)$, $||\mu_\lambda|| \leq 1$, we get

(4) $\quad ||\mu * f|| \leq ||f||$ for each $f \in L^1(\omega)$.

Given $\eta > 0$, let $\psi \in C_0(\omega)$ with $||\psi|| = 1$ be such that

(5) $\quad ||\mu|| < |\langle\mu, \psi\rangle| + \eta$

Now, for this ψ, as in Lemma 1.3.3, let $f \in L^1(\omega)$ with $||f|| = 1$ be such that $|\langle\mu * f, \psi\rangle - \langle f, \psi\rangle| < \eta$. Thus, by (4) and (5)

$$||\mu|| \leq |\langle\mu, \psi\rangle| + \eta \leq |\langle\mu * f, \psi\rangle| + 2\eta \leq ||\mu * f|| + 2\eta$$
$$\leq ||f|| + 2\eta = 1 + 2\eta$$

and this completes the proof of the claim. So far we have proved that

$$\mathrm{co}[TE_K : so] \subset \{\mu : \mu \in M(\omega),\ ||\mu|| \leq 1,\ s(\mu) \subset K\}$$

To prove the inverse inclusion, from lemma 1.3.3 it follows that the identity from $\mathrm{co}[TE_K : so]$ into $[M(\omega) : \sigma]$ is continuous. Thus $\mathrm{co}[TE_K : so]$ is σ - compact and hence must contain $\mathrm{co}[TE_K : \sigma]$. On the other hand the set $\{\mu \in M(\omega) : ||\mu|| \leq 1,\ s(\mu) \subset$ is σ compact and the set of the extreme points of it is TE_K thus by the Krein-Millman theorem

$$\mathrm{co}[TE_K : \sigma] = \{\mu \in M(\omega) : ||\mu|| \leq 1,\ s(\mu) \subset K\}$$
$$\supset \mathrm{co}[E_K : so] \supset \mathrm{co}[TE_K : \sigma]$$

and this gives the result.

1.3.6 Lemma. $\mathrm{co}[TE_{R^+} : so]$ is the unit ball in $M(\omega)$.

Proof. Let $\mu \in M(\omega)$, $||\mu|| \leq 1$, and let $K_n = [0, n]$ $(n \in \mathbb{N})$. Then $\mu_n = \mu|K_n \in M(\omega)$ is such that $||\mu_n|| \leq 1$, $\mu_n \in \mathrm{co}[TE_{K_n} : so]$, and $\mu_n \overset{||\cdot||}{\to} \mu$. Thus μ is in the norm closure of $\bigcup_{n=1}^{\infty} co[TE_{K_n} : so]$, which lies within $co[TE_{R^+} : so]$.

Next lemma shows that the bounded approximate identity $\{g_n : n \in N\}$ is a bounded σ-approximate identity for $M(\omega)$. More precisely:

1.3.7 Lemma. For every $\mu \in M(\omega)$, $\mu * g_n \overset{\sigma}{\to} \mu$.

Proof. First let μ have a compact support then for each n, $\mu * g_n$ has a compact support, moreover, since,

$$S(\mu * g_n) \subset S(\mu) + S(g_n) \quad \text{and} \quad S(g_n) \subset [0, 1] \text{ we have}$$

$$S(\mu * g_n) \subset S(\mu) + [0, 1] .$$ Now, $\mu * g_n \overset{so}{\to} \mu$ and $||\mu * g_n|| \le ||\mu||$. Therefore, by lemma 1.3.3 $\mu * g_n \overset{\sigma}{\to} \mu$. For a general μ , according to proposition 1.1.12, the σ-limit of $\mu * e_n$ is $\overline{1}(\mu)$, where 1 is the identity operator on $L^1(\omega)$ and $\overline{1}$ is its extension as in proposition 1.1.12 to $M(\omega)$. Thus, continuity of $\overline{1}$ implies that $\overline{1}(\mu) = \lim \overline{1}(\mu_n) = \lim \mu_n = \mu$ (the limits are all norm limits), and this completes the proof.

1.3.8 Remark. If $\{f_\lambda : \lambda \in \Lambda\}$ is any bounded approximate identity then for $\mu \in M(\omega)$ by proposition 1.1.12 the σ-limit, $\lim(\mu * f_\lambda)$ does not depend on $\{f_\lambda : \lambda \in \Lambda\}$ and is $\overline{1}(\mu)$ which by the above lemma is equal to μ .

Lemma 1.3.9 If θ is an isometric isomorphism of $L^1(\omega)$, then $\{\theta(g_n) : n \in \underset{\sim}{N}\}$ is a bounded approximate identity for $L^1(\omega)$.

Proof. For $f \in L^1(\omega)$, we have

$$||f - f*\theta(g_n)|| = ||\theta(\theta^{-1}f - \theta^{-1}f*g_n)|| = ||\theta^{-1}f - \theta^{-1}f*g_n|| \to 0$$

Now we are ready to extend every isometric isomorphism of $L^1(\omega)$ to an isometric isomorphism of $M(\omega)$. Our next lemma is in fact a corollary to proposition 1.1.12 and lemma 1.3.6.

Lemma 1.3.10 If θ is an isometric isomorphism of $L^1(\omega)$, then θ has an extension to an isometric isomorphism $\bar{\theta}$ of $M(\omega)$ moreover $\bar{\theta}$ is continuous from $[M(\omega),bso]$ into $[M(\omega),\sigma]$.

Proof. By proposition 1.1.12, the map $\bar{\theta}$ defined by

$$\bar{\theta}(\mu) = \lim_{\sigma} \theta(\mu * g_n) \qquad (\mu \in M(\omega))$$

is an extension of θ to $M(\omega)$. Now we show that it is onto, 1-1 and isometric. To prove that $\bar{\theta}$ is onto let $\mu \in M(\omega)$ and let $\nu = \lim_{\sigma} \theta^{-1}(\mu * g_n)$. Then,

$$\begin{aligned}
\bar{\theta}(\nu) &= \lim_{k} \theta(\lim_{j} \theta^{-1}(\mu * g_j) * g_k) \\
&= \lim_{k} \theta(\lim_{j} \theta^{-1}(\mu * g_j) * \theta^{-1}\theta g_k) = \lim_{k} \theta(\lim_{j}(\theta^{-1}(\mu * g_j) * \theta^{-1}\theta g_k)) \\
&= \lim_{k} \theta(\lim_{j}(\theta^{-1}(\mu * \theta g_k * g_j))) = \lim_{k} \theta(\lim_{j}(\theta^{-1}(\mu * \theta g_k) * \theta^{-1}(g_j))) \\
&= \lim_{k} \theta(\theta^{-1}(\mu * \theta g_k)) = \lim_{k} \mu * \theta(g_k) = \mu
\end{aligned}$$

all the limits are σ-limit, and we have used lemma 1.3.7, remark 1.3.8 and lemma 1.3.9. Thus $\bar{\theta}$ is onto. To show that it is 1-1 let $\bar{\theta}(\mu) = 0$, then

$$\bar{\theta}(\mu * g_n) = \bar{\theta}(\mu) * \bar{\theta}(g_n) = 0 \qquad (n \in N)$$

But $\mu * g_n \in L^1(\omega)$. Thus, $\theta(\mu * g_n) = \bar{\theta}(\mu * g_n) = 0$ $(n \in \underset{\sim}{N})$. Since θ is 1-1 , we get $\mu * g_n = 0$ $(n \in \underset{\sim}{N})$. Thus $\mu = \lim_{\sigma}(\mu * g_n) = 0$ and $\bar{\theta}$ is 1-1 . To prove $\bar{\theta}$ is isometric, from the definition of $\bar{\theta}$ and σ compactness of the unit ball of $M(\omega)$ we obtain $\|\bar{\theta}(\mu)\| \le \|\mu\|$. On the other hand the argument used to show that $\bar{\theta}$ is onto implies that the inverse of $\bar{\theta}$ can be defined by,

$$(\bar{\theta})^{-1}(\mu) = \lim_{\sigma} (\theta^{-1}(\mu)*e_j) = \overline{(\theta^{-1})}(\mu) \qquad (\mu \in M(\omega))$$

Thus, since θ^{-1} is an isometric isomorphism of $L^1(\omega)$, as above, we have,

$$||(\bar{\theta})^{-1}(\mu)|| = ||\overline{(\theta^{-1})}(\mu)|| \leq ||\mu||$$

Thus $\bar{\theta}$ is an isometric isomorphism.

1.3.11 Theorem. If θ is an isometric isomorphism of $L^1(\omega)$, then there is a number $\alpha \in R$, such that for every $f \in L^1(\omega)$, we have

$$(\theta f)(x) = e^{i\alpha x} f(x) \qquad (x \in R^+)$$

Proof. By 1.3.10, θ has an extension to an isometric isomorphism of $M(\omega)$. Given $x \in R^+$, then $\bar{\theta}(\frac{1}{\omega(x)}\delta_x)$ must be an extreme point of the unit ball of $M(\omega)$. Thus, for every $x \in R^+$ we have

$$(1) \quad \bar{\theta}\left(\frac{1}{\omega(x)}\delta_x\right) = \frac{\gamma(x)}{\omega(\alpha(x))}\delta_{\alpha(x)} \quad \text{with } |\gamma(x)| = 1 \text{ and } \alpha(x) \in R^+$$

If we apply $\bar{\theta}$ to both sides of the equation $\delta_x * \delta_y = \delta_{x+y}$ $(x, y \in R^+)$ we obtain

$$(2) \quad \frac{\gamma(x+y)\omega(x+y)}{\omega(\alpha(x+y))}\delta_{\alpha(x+y)} = \frac{\gamma(x)\omega(x)}{\omega(\alpha(x))}\delta_{\alpha(x)} * \frac{\gamma(y)\omega(y)}{\omega(\alpha(y))}\delta_{\alpha(y)} \quad (x, y \in R^+)$$

From (2) we obtain

$$(3) \quad \frac{\gamma(x+y)\omega(x+y)}{\omega(\alpha(x+y))} = \frac{\gamma(x)\omega(x)}{\omega(\alpha(x))} \; \frac{\gamma(y)\omega(y)}{\omega(\alpha(y))} \qquad \text{and}$$

$$\alpha(x+y) = \alpha(x) + \alpha(y)$$

This shows that the function ϕ defined by $\phi(x) = \frac{\gamma(x)\omega(x)}{\omega(\alpha(x))}$ $(x \in R^+)$

is multiplicative on R^+. Now, we show that the functions γ and α are continuous. Let $f \in C_0(\omega)$, with $f(x) > 0$ $(x \in R^+)$, we have

(4) $$\left|(\bar{\theta}(\frac{1}{\omega(x)}\delta_x))(f)\right| = \frac{1}{\omega(\alpha(x))}\delta_{\alpha(x)}(f) .$$

Consequently,

(5) $$\gamma(x)\left|\frac{1}{\omega(x)}(\bar{\theta}(\delta_x))(f)\right| = (\bar{\theta}(\frac{1}{\omega(x)}\delta_x))(f)$$

Thus, to show that the map $x \to \gamma(x)$ is continuous, it suffices that we show the map $x \to (\bar{\theta}(\frac{1}{\omega(x)}\delta_x))(f)$ is continuous from R^+ into the complex numbers. The map $x \to \frac{1}{\omega(x)}\delta_x$ is continuous from R^+ to $[E_{R^+} : so]$ by Lemma 1.3.4 and the map $\frac{1}{\omega(x)}\delta_x \to (\bar{\theta}(\frac{1}{\omega(x)}\delta_x))(f)$ from $[E_{R^+} : so]$ into $\underset{\sim}{C}$ is continuous by proposition 1.1.12 (II). Thus $x \to \gamma(x)$ $(x \in R^+)$ is continuous. The continuity of α follows by considering the maps

(5) $$x \to \frac{1}{\omega(x)}\delta_x \to \bar{\theta}(\frac{1}{\omega(x)}\delta_x) = \frac{\gamma(x)}{\omega(\alpha(x))}\delta_{\alpha(x)} \to \frac{1}{\omega(\alpha(x))}\delta_{\alpha(x)} \to \alpha(x)$$

where the continuity of the first and the last maps follow from lemma 1.3.4. Since the function α is continuous and additive there is a positive a, such that $\alpha(x) = ax$ $(x \in R^+)$. By the continuity of γ and ω and α the function $\frac{\gamma(x)\omega(x)}{\omega(\alpha(x))}$ is continuous, and since it is multiplicative there is $b \in \underset{\sim}{C}\backslash\{0\}$ such that

(6) $$\frac{\gamma(x)\omega(x)}{\omega(ax)} = b^x \qquad (x \in R^+)$$

Using (6) we prove that $a = 1$. Considering the absolute value of both sides in (6) we get

(7) $$\frac{\omega(x)}{\omega(ax)} = |b|^x \qquad (x \in R^+)$$

or

$$\omega(x) = |b|^{x}\omega(ax) \tag{8}$$

From (8) it follows

$$\omega(ax) = |b|^{ax}\omega(a^2x) \tag{9}$$

From (8) and (9) we obtain

$$\omega(x) = |b|^{x+ax}\omega(a^2x) \tag{10}$$

Following this pattern by considering $\omega(a^kx)$ $(k = 1, 2, \ldots, n-1)$ we obtain

$$\omega(x) = |b|^{x+ax+a^2x + \ldots + a^{n-1}x}\,\omega(a^nx) = |b|^{\frac{a^n-1}{a-1}x}\,\omega(a^nx) \tag{11}$$

From (11) it follows,

$$-\frac{1}{a^nx}\log\omega(x) = \frac{-1}{a^n}\,\frac{a^n-1}{a-1}\log|b| - \frac{1}{a^nx}\log\omega(a^nx) \tag{12}$$

therefore, if $a > 1$, we let $n \to \infty$ in both sides of (12) and use the fact that $L^1(\omega)$ is radical to obtain $0 = -\frac{1}{a-1}\log|b| + \infty$ which is a contradiction. On the other hand if $a < 1$, then from (11), by letting $n \to \infty$ we obtain

$$\omega(x) = |b|^{\frac{-x}{a-1}}\omega(0) = |b|^{\frac{-x}{a-1}} \tag{13}$$

and this contradicts the fact that $L^1(\omega)$ is a radical algebra. Thus $a = 1$ and the equation (6) gives

$$\gamma(x) = b^x \qquad (x \in R^+) \tag{11}$$

since $|\gamma(x)| = 1$, there is an $\alpha \in R$ such that $\gamma(x) = e^{i\alpha x}$ $(x \in R^+)$. So far we have proved that there is an $\alpha \in R$ such that

$$\bar{\theta}\left(\frac{1}{\omega(x)}\,\delta_x\right) = \frac{e^{i\alpha x}}{\omega(x)}\,\delta_x \qquad (x \in R^+) \tag{13}$$

Now, let the mapping $T : M(\omega) \to M(\omega)$ be defined by

$$(14) \qquad d(T\mu)(x) = e^{i\alpha x} d\mu(x)$$

It is easy to see that T is an isometric isomorphism of $M(\omega)$. The two operators $\overline{\theta}$ and T coincide on E_{R^+}. By Lemma 1.3.6 every $\mu \in M(\omega)$ is so-limit of a bounded net $\{\mu_j : j \in J\}$ where each μ_j is a linear combination of measures of the form $\frac{1}{\omega(x)} \delta_x$ $(x \in R^+)$ and $||\mu_j|| \le ||\mu||$ $(j \in J)$. Now by proposition 1.1.12 (II) we have

$$(15) \qquad \overline{\theta}(\mu) = \lim_{\sigma} \overline{\theta}(\mu_j) = \lim_{\sigma} T(\mu_j) .$$

The operator T leaves $L^1(\omega)$ invariant and is invertible, therefore for every $f \in L^1(\omega)$, $T^{-1}f \in L^1(\omega)$ and

$$(16) \qquad T(\mu_j)*f = T(\mu_j*T^{-1}f) \overset{||\cdot||}{\to} T(\mu*T^{-1}f) = T(\mu)*f$$

Thus $T(\mu_j) \overset{so}{\to} T(\mu)$. Since $||T(\mu_j)|| = ||\mu_j|| \le ||\mu||$ another application of proposition 1.1.12 (II) imply that $T(\mu_j) \overset{\sigma}{\to} T(\mu)$ and this together with (15) imply $\overline{\theta}(\mu) = T(\mu)$ $(\mu \in M(\omega))$. In particular the restriction of $\overline{\theta}$ to $L^1(\omega)$, θ is given by

$$(\theta f)(x) = e^{i\alpha x} f(x) \qquad (f \in L^1(\omega),\ x \in R^+ \text{ a.e})$$

and this proves the theorem.

The method used in this chapter shows that if $L^1(\omega_1)$ and $L^1(\omega_2)$ are two radical algebras and if there exists an isometric isomorphism from $L^1(\omega_1)$ onto $L^1(\omega_2)$, then similar to the formula (7) of theorem 1.3.11, there exist $a > 0$, $b > 0$, such that

$$\frac{\omega_1(x)}{\omega_2(ax)} = b^x \qquad (x \in R^+)$$

Conversely, if a and b with above property exist then the map $\theta : L^1(\omega_1) \to L^1(\omega_2)$, defined by

$$(\theta f)(x) = \frac{1}{a} f\left(\frac{x}{a}\right) b^{\frac{x}{a}} \qquad (x \in R^+)$$

is an isometric isomorphism form $L^1(\omega_1)$ onto $L^1(\omega_2)$. Thus we have:

1.3.12 Theorem. A necessary and sufficient condition for two radical algebras $L^1(\omega_1)$ and $L^1(\omega_2)$ to be isometrically isomorphic is the existence of $a > 0$, $b > 0$, such that

$$\frac{\omega_1(x)}{\omega_2(ax)} = b^x \qquad (x \in R^+)$$

CHAPTER 1.4

Derivations on $L^1(\omega)$

1.4.1 In this chapter we study the derivations on the algebras $L^1(\omega)$. By definition a derivation on an algebra A with sum $+$ and product $\circ$ is a linear mapping D, which satisfies

(1) $$D(x.y) = D(x).y + x.D(y) \qquad (x, y \in A)$$

When a commutative Banach algebra A is semi-simple then 0 is the only derivative on A [Johnston cf.18]. Thus, for semi-simple $L^1(\omega)$, 0 is the only derivation and in the rest of this chapter we will assume that $L^1(\omega)$ is radical. It is natural if we ask whether there are non-zero derivations on $L^1(\omega)$, when it is radical. We characterize all weights ω, for which, the corresponding radical algebra has a non-zero derivation and find the general form and the norm of these derivations. Luckily, every derivation on $L^1(\omega)$ is continuous this is a corollary of a more general result of Jewelland Sinclair which we state.

1.4.2 Theorem. If B is a commutative Banach algebra with the property that for each infinite dimensional closed ideal J in B there is a $b \in B$ such that $J \supset (Jb)^-$, and if B contains no non-zero finite dimensional nilpotent ideal then every derivation on B is continuous [cf.17 Remark 3(a)]. The inclusion in the above theorem is strict.

Now, we prove that the algebras $L^1(\omega)$ satisfy the hypothesis of theorem 1.4.2. But first we need the following definition and theorems.

1.4.3 Definition. We denote by L^1_{loc} the space of all Lebesgue measurable functions which are locally integrable, i.e. $f \in L^1_{loc}$

if and only if $\int_K |f(x)|dx < \infty$ for every compact subset K of R^+. With the usual pointwise addition of functions and scalar multiplication and convolution defined as in (1.1.1)(3), L^1_{loc} is an algebra. Obviously for every weight function ω, $L^1(\omega)$ is a subalgebra of L^1_{loc}. For every $f \in L^1_{loc}\setminus\{0\}$ let $\alpha(f)$ = infimum of the support of f, then we have the following theorem, known as the Titchmarsh's Convolution theorem.

1.4.4 Theorem [Titchmarsh] If $f, g \in L^1_{loc}$ and $f*g = 0$, then $f = 0$ or $g = 0$.

Proof [cf. 36 Th.152 p.325].

We also have

1.4.5 Theorem. If $f, g \in L^1_{loc}\setminus\{0\}$, then

$$\alpha(f*g) = \alpha(f) + \alpha(g)$$

Proof. For a proof due to G.R. Allan see [cf.9, Th.7.4].

From theorem 1.4.4 it follows that the algebra L^1_{loc} and its subalgebras are integral domains

1.4.6 Corollary. Derivations on $L^1(\omega)$ are continuous.

Proof. The algebra $L^1(\omega)$ does not contain a nilpotent ideal since it is an integral domain. If J is an infinite dimensional closed ideal let $f \in L^1(\omega)$, with $\alpha(f) = 1$. Then if a = infimum of support $\{g : g \in J\}$ by theorem 1.4.5 we have, infimum of support $\{h : h \in \overline{J*f}\} = a + 1$. Thus $J \supset \overline{J*f}$, and by theorem 1.4.2 every derivation on $L^1(\omega)$ is continuous.

To characterize the derivations of $L^1(\omega)$, given a derivation D on $L^1(\omega)$ we will extend it to a derivation $\overline{D}$ on $M(\omega)$, and at the same time bearing in mind that $\overline{D}$ is an extension of a derivation on $L^1(\omega)$, we will find $\overline{D}$. We use the notation of chapter 1.3. In the next lemma we let

$$g_n = \frac{e_n}{\|e_n\|}, \quad \text{where} \quad e_n = n\chi_{[0,1/n]} \quad (n = 1, 2, \ldots) .$$

1.4.7 Lemma. If D is a derivation on $L^1(\omega)$, then $D(g_n) \overset{\sigma}{\to} 0$.

Proof. Since D is continuous, $D(g_n)$ is bounded, thus the σ-compactness of the unit ball of $M(\omega)$ implies that there is a σ-limit point λ and a subsequence $\{g_{n_k}\}$ such that $D(g_{n_k}) \overset{\sigma}{\to} \lambda$. We have

$$(1) \qquad D(g_{n_k} * f) = D(g_{n_k}) * f + g_{n_k} * D(f) \qquad (f \in L^1(\omega))$$

Now, if we find the σ-limit of both sides as $k \to \infty$, by lemmas 1.1.10 and 1.3.7 we get

$$(3) \qquad D(f) = \lambda * f + D(f) \qquad (f \in L^1(\omega))$$

Thus $\lambda * f = 0$ for every $f \in L^1(\omega)$. In particular,

$$(3) \qquad \lambda * g_n = 0 \qquad (n \in N)$$

Again an application of lemma 1.3.7 gives $\lambda = 0$. Thus $\lim_\sigma D(g_n) = 0$.

1.4.8 Lemma. If D is a derivation on $L^1(\omega)$, then for every $\mu \in M(\omega)$, the limit $\lim_\sigma D(g_n * \mu)$ exists and the map $\overline{D} : M(\omega) \to M(\omega)$, defined by

$$\overline{D}(\mu) = \lim_\sigma D(g_n * \mu) \qquad (\mu \in M(\omega))$$

is a norm preserving extension of D.

Proof. As in the proof of lemma 1.4.7 the boundedness of $D(g_n*\mu)$ and the σ-compactness of the unit ball of $M(\omega)$ imply that $D(g_n*\mu)$ has a σ-limit point λ_μ, and there is a subnet $\{g_{n_k}\}$ such that $\lim_\sigma D(g_{n_k}*\mu) \to \lambda_\mu$. Given $f \in L^1(\omega)$, by lemma 1.1.10 we have

$$(1) \quad D(g_{n_k}*\mu*f) - g_{n_k}*\mu*Df = D(g_{n_k}*\mu)*f \overset{\sigma}{\to} \lambda_\mu*f \qquad (f \in L^1(\omega))$$

On the other hand,

$$(2) \quad D(g_{n_k}*\mu*f) - g_{n_k}*\mu*Df \overset{\sigma}{\to} D(\mu*f) - \mu*D(f) \qquad (f \in L^1(\omega))$$

By comparing (1) and (2) we obtain

$$(3) \quad D(\mu*f) - \mu*Df = \lambda_\mu*f \qquad (f \in L^1(\omega))$$

Now let in (3) $f = g_n (n = 1, 2, \ldots)$ and find the σ-limit of both sides by lemma 1.4.7 we obtain $\lim_\sigma D(\mu*g_n) = \lambda_\mu$. It is easy to verify that $\overline{D}$ is an extension of D. To prove that $\overline{D}$ is a derivation first let $\mu \in M(\omega)$ and $f \in L^1(\omega)$, then,

$$(4) \quad \overline{D}(\mu*f) = \lim_\sigma D(\mu*f*g_n) = \lim_\sigma (D(\mu*g_n)*f + D(f)*\mu*g_n)$$

$$= \overline{D}(\mu)*f + \overline{D}f*\mu$$

[by lemmas 1.1.10 and 1.4.7]. Next, for $\mu, \nu \in M(\omega)$, we have

$$(5) \qquad \overline{D}(\mu*\nu) = \lim_\sigma D(\mu*\nu*g_n)$$

The left hand side of (5) by (4) is equal to

$$(6) \quad \lim_\sigma[\overline{D}(\mu)*(\nu*g_n) + \mu * \overline{D}(\nu*g_n)] = \overline{D}(\mu)*\nu + \mu*\overline{D}(\nu) .$$

Finally, since $\|g_n\| = 1$, $(n \in \underset{\sim}{N})$ we have

$\|\overline{D}(\mu)\| = \|\lim_\sigma D(\mu*g_n)\| \le \|D\| \cdot \|\mu\|$. Thus $\|\overline{D}\| = \|D\|$.

1.4.9 Notation. Given $\mu \in M(\omega)$, let $\alpha(\mu)$ be the infimum of the support of μ .

1.4.10 Lemma. If $\mu, \nu \in M(\omega)$, then $\alpha(\mu*\nu) \geq \alpha(\mu) + \alpha(\nu)$.

Proof. We obviously have $s(\mu*\nu) \subset s(\mu) + s(\nu)$ therefore

$$\alpha(\mu*\nu) \geq \alpha(\mu) + \alpha(\nu) .$$

The proof of the next lemma is from [11].

1.4.11 Lemma. For each $a \in R^+$, $\alpha(D(\delta_a)) \geq a$.

Proof. For each natural number n , let

$$b = \frac{a}{n}, \quad \text{then} \quad \delta_a = \delta_{nb} = (\delta_b)^n$$

$$D(\delta_a) = D(\delta_{nb}) = D(\delta_{(n-1)b}*\delta_b) = D(\delta_{(n-1)b})*\delta_b + \delta_{(n-1)b}*D(\delta_b)$$

Therefore by induction

$$D(\delta_a) = n\delta_{(n-1)b}*D(\delta_b)$$

Thus,

$$\alpha(D\delta_a) \geq \alpha(n\delta_{(n-1)b}) + \alpha(D(\delta_b)) \geq (n-1)b = \frac{n-1}{n} a$$

Since n is arbitrary we get $\alpha(D\delta_a) \geq a$.

1.4.12 Definition. A complex measure μ defined on the σ-algebra of Borel sets of R^+ is called locally finite, if it is of finite variation on every compacta.

1.4.13 Lemma. If D is a derivation on $M(\omega)$ then there is a complex regular Borel measure μ which is locally finite on R^+ such that

(I) $\displaystyle \sup_{t \in (0,\infty)} \frac{t}{\omega(t)} \int_0^\infty \omega(t+s)\, d|\mu|(s) = K < \infty$

(II) $D(\delta_a) = a\mu*\delta_a \qquad (a \in R^+)$

Proof. Equation (II) is obviously true for $a = 0$. Let $a > 0$, since $\alpha(D\delta_a) \geq a$, the measure $\frac{1}{a}(D\delta_a)*\delta_{-a}$ is a locally finite regular complex Borel measure on R^+, we denote this measure by μ_a. Thus

(1) $$D(\delta_a) = a\mu_a * \delta_a \qquad (a > 0)$$

We prove that for every $a, b > 0$, $\mu_a = \mu_b$. For every $n \in \underset{\sim}{N}$, by (1) we have

(2) $$D(\delta_{na}) = na\, \mu_{na} * \delta_{na}$$

and by induction and (1)

(3) $$D(\delta_{na}) = n\delta_{(n-1)a}*D(\delta_a) = na\, \delta_{(n-1)a}*\mu_a*\delta_a = na\delta_{na}*\mu_a,$$

By comparing (2) and (3) we obtain $\mu_{na} = \mu_a$ $(n \in \underset{\sim}{N},\ a > 0)$
Thus, for every $m, n \in \underset{\sim}{N}$ we have

(4) $$\mu_{\frac{n}{m}} = \mu_{\frac{1}{m}} = \mu_{m.\frac{1}{m}} = \mu_1$$

Thus $\mu_r = \mu_s$ if r, s are any two positive rationals and we denote the common measure μ_r by μ. Thus we have

(5) $$D(\delta_r) = r\mu*\delta_r \qquad (r \in \overset{+}{Q})$$

Now, let $a > 0$ be irrational and let $r_n \uparrow a$, $r_n \in Q^+$, then by (5) we have

(6) $$D(\delta_{r_n}) = r_n\mu*\delta_{r_n}$$

Now, we show that as $n \to \infty$, the left hand side of (6) tends to $D(\delta_a)$ and the right hand side of (6) tends to $a\mu*\delta_a$ in the topology $'so$. Given $f \in L^1(\omega)$, we have

(7) $D(\delta_{r_n})*f = D(\delta_{r_n}*f) - \delta_{r_n}*D(f) \overset{\|\cdot\|}{\rightarrow} D(\delta_a*f) - \delta_a*D(f) = D(\delta_a)*f$

Thus, $D(\delta_{r_n}) \overset{so}{\rightarrow} (D\delta_a)$.

Now we look at the right hand side of (6). The measure μ is a locally finite measure which may or may not belong to $M(\omega)$ [see 1.4.22]. However, equation (5) shows that for every $s \in Q^+$, $\mu*\delta_s \in M(\omega)$. Let $s \in Q^+$, be as small as $r_n - s$ is positive for large values of n . Then we write the right hand side of (6) as $r_n(\mu*\delta_s)*\delta_{r_n-s}$, since $\mu*\delta_s \in M(\omega)$ and $r_n - s \to a - s$

$$r_n(\mu*\delta_s)*\delta_{r_n-s} \overset{so}{\rightarrow} a(\mu*\delta_s)*\delta_{a-s} = a\mu*\delta_a .$$

Since D is continuous,

$$\frac{\|D(\delta_a)\|}{\|\delta_a\|}$$

is bounded by $\|D\|$. Thus

$$\frac{\|a\mu*\delta_a\|}{\|\delta_a\|} = \frac{a\int_0^\infty \omega(a+s)d|\mu|(s)}{\omega(a)} \leq \|D\|$$

and this completes the proof of lemma.

1.4.14 Notation. If ν is a locally finite measure then we define $t\nu$ to be a measure defined by $(t\nu)(E) = \int_E x d\nu(x)$. If $f \in L^1(\omega)$, then by this definition $(tf)(x) = xf(x)$.

1.4.15 Lemma. Let μ be a locally finite measure on R^+ such that

$$\sup_{a\in(0,\infty)} \frac{a\int_0^\infty \omega(a+s)d|\mu|(s)}{\omega(a)} < \infty$$

then for every $f \in L^1(\omega)$, $tf*\mu \in L^1(\omega)$ and the map D_1 defined by

$$(1) \qquad D_1(f) = tf*\mu \qquad (f \in L^1(\omega))$$

is a derivation on $L^1(\omega)$.

Proof. For $f \in L^1(\omega)$, we have

$$(2) \quad \int_0^\infty |(tf*\mu)(x)|\, dx = \int_0^\infty \left|\int_0^x (x-y)f(x-y)d\mu(y)\right|\omega(x)dx$$

$$\leq \int_0^\infty \int_0^x |x-y||f(x-y)|d|\mu|(y)\omega(x)dx$$

Now let ψ be a function on $R^+ \times R^+$ defined by

$$\psi(x, y) = \begin{cases} |x-y||f(x-y)| & (y < x) \\ 0 & \text{elsewhere} \end{cases}$$

Then the right hand side of (2) is equal to $\int_{R^+}\int_{R^+} \psi(x, y)d|\mu|(y)dx$ and by Fubini's theorem this is equal to

$$\int_{R^+}\int_{R^+} \psi(x,y)\omega(x)dx\, d|\mu|(y) = \int_0^\infty \int_y^\infty (x-y)|f(x-y)|\omega(x)dxd|\mu|(y)$$

By a change of variable $x = y + z$ the above integral becomes

$$\int_0^\infty \int_0^\infty z\omega(y+z)|f(z)|dzd|\mu|(y) = \int_0^\infty \omega(z)|f(z)| \frac{z}{\omega(z)} \int_0^\infty \omega(y+z)d|\mu|(y)dz$$

$$\leq ||f|| \quad K .$$

Thus $D_1(f) \in L^1(\omega)$. Now, we show that D_1 is a derivation on $L^1(\omega)$. For $f, g \in L^1(\omega)$ we have

$$(D(f)*g)(x) + (f*D(g))(x) = \int_0^x \int_0^{x-y} (x-y-z)f(x-y-z)d\mu(z)g(y)dy$$

$$+ \int_0^x f(x-y) \int_0^y (y-z)g(y-z)d\mu(z)dy = \int_0^x \int_0^{x-y} (x-z)f(x-y-z)g(y)d\mu(z)dy$$

$$- \int_0^x y \int_0^{x-y} f(x-y-z)g(y)d\mu(z)dy + \int_0^x f(x-y) \int_0^y (y-z)g(y-z)d\mu(z)dy$$

Again by Fubini's theorem the first integral in the above sum is equal to

$$\int_0^x \int_0^{x-z} (x-z) f(x-y-z) g(y) dy \, d\mu(z)$$

and by the change of variable $y = t + z$ in the third integral, (after using Fubini's theorem) the last two integrals cancel and we get

$$(D(f)*g)(x) + (f*Dg)(x) = \int_0^x \int_0^{x-z} (x-z) f(x-y-z) g(y) d\mu(z) = D(f*g)(x) .$$

1.4.16 Note. If $\overline{D}_1$ is the extension of D_1 as in 1.4.8 then

$$\overline{D}_1(\nu) = t\nu*\mu \qquad (\nu \in M(\omega))$$

This is because, similar to what we did in 1.4.15 the map $D_2(\nu) = t\nu*\mu (\nu \in M(\omega))$ defines a derivation on $M(\omega)$ and $\overline{D}_1$ and D_2 coincide on $L^1(\omega)$. Since $L^1(\omega)$ is so dense in $M(\omega)$, we have $\overline{D}_1 = D_2$.

1.4.17 Theorem. If D is a derivation on $L^1(\omega)$, then there is a locally finite measure μ on R^+, such that

$$(1) \qquad \sup_{x \in (0,\infty)} \frac{x}{\omega(x)} \int_0^\infty \omega(x+y) \, d|\mu|(y) < \infty$$

$$Df = tf*\mu \qquad (f \in L^1(\omega))$$

Proof. By Lemma 1.4.8, D has an extension to a derivation $\overline{D}$ of $M(\omega)$. By lemma 1.4.13 corresponding to $\overline{D}$ there is a locally finite measure μ such that

$$\overline{D}(\delta_a) = a\mu*\delta_a$$

$$\sup_{t \in (0,\infty)} \frac{t}{\omega(t)} \int_0^\infty \omega(t+s) \, d|\mu|(s) < \infty$$

Now let Δ be a map on $M(\omega)$ defined by

$$\Delta(\nu) = \overline{D}(\nu) - t\nu*\mu \qquad (\nu \in M(\omega)) .$$

Then Δ is a derivation on $M(\omega)$ and we have

$$\Delta(\delta_a) = a\delta_a*\mu - a\delta_a*\mu = 0 \qquad (a \in R^+) .$$

By lemma 1.3.6 every $\nu \in M(\omega)$ is so-limit of net $\{\nu_j : j \in J\}$ where each ν_j is a linear combination of measures of the form $\frac{1}{\omega(x)} \delta_x$ $(x \in R^+)$. Thus $\Delta(\nu) = 0$ $(\nu \in M(\omega))$. Therefore $\overline{D}(\nu) = t\nu*\mu$ $(\nu \in M(\omega))$. In particular for $f \in L^1(\omega)$, we have $D(f) = tf*\mu$.

1.4.18 Corollary.

$$||D|| = \sup_{x\in(0,\infty)} \frac{x}{\omega(x)} \int_0^\infty \omega(x+y)\, d|\mu|(y) .$$

Proof. From the proof of lemma 1.4.15 it follows that

$$(1) \qquad ||D|| \le \sup_{x\in(0,\infty)} \frac{x}{\omega(x)} \int_0^\infty \omega(x+y)\, d|\mu|(y)$$

If $\overline{D}$ is the extension of D , we saw in lemma 1.4.8 that $||\overline{D}|| = ||D||$, but,

$$(2) \quad ||D|| = ||\overline{D}|| \ge \sup_{x\in(0,\infty)} \frac{||D\delta_x||}{||\delta_x||} = \sup_{x\in(0,\infty)} \frac{x}{\omega(x)} \int_0^\infty \omega(x+y)\, d|\mu|(y)$$

From (1) and (2) the result follows.

1.4.19 Note. Theorem 1.4.17 states that if D is a derivation on $L^1(\omega)$, then it must be of that special form. But then the measure μ might be zero and there might be no non-zero derivations, this is the case for some weights as we shall see. In the next theorem we give a necessary and sufficient condition for the weight ω, under which the algebra $L^1(\omega)$ has a non-zero derivation.

1.4.20 Theorem. A necessary and sufficient condition for $L^1(\omega)$ to have a non-zero derivation is that there exist a positive number b such that

$$\sup_{a\in(0,\infty)} a\,\frac{\omega(a+b)}{\omega(a)} < \infty$$

Proof. If the number b with $\sup_{a\in(0,\infty)} a\,\frac{\omega(a+b)}{\omega(a)} < \infty$ exists, then the map $D(f) = tf*\delta_b$ $(f \in L^1(\omega))$ is a derivation on $L^1(\omega)$. Conversely, suppose that D is a non-zero derivation on $L^1(\omega)$ and μ is the measure that corresponds to D as in theorem 1.4.17. Then $\mu \neq 0$. Also $\mu \neq \delta_0$, since δ_0 does not satisy (1) of theorem 1.4.17. Thus there exist, b, c with $0 < b < c$ such that $|\mu|[b,c] \neq 0$. We have

$$(1) \quad \|D\| = \sup\left\{a\int_0^\infty \frac{\omega(a+s)}{\omega(a)}\,d|\mu|(s) \;:\; a > 0\right\}$$

$$\geq a\int_b^c \frac{\omega(a+s)}{\omega(a)}\,d|\mu|(s) \qquad (a > 0)$$

Now, let $K = \sup\{\omega(c - s) : s \in [b, c]\} < \infty$, then

$$(2) \quad \omega(a + c) \leq \omega(a + s)\omega(c - s) \leq K\omega(a + s) \quad (a > 0,\ s \in [b, c])$$

Hence,

$$\|D\| \geq a\,\frac{\omega(a+c)}{\omega(a)}\ \frac{1}{K}\ |\mu|[b, c] \qquad (a > 0)$$

and the result follows.

1.4.21 Examples. The weights $\omega_1(t) = e^{-t^2}$ and $\omega_2(t) = e^{-t \log t}$ both satisfy the hypothesis of theorem 1.4.20. For ω_1, b can be any positive number, while for ω_2, b can be any positive number not less than 1. For the weight $\omega_3(t) = e^{-t \log \log t}$, for every $b > 0$ we have

$$\sup_{t>0} t \frac{\omega(t+b)}{\omega(t)} = \sup t \frac{e^{-(t+b)\log\log(t+b)}}{e^{-t \log \log t}} = \infty$$

Therefore $L^1(\omega_3)$ does not have non-zero derivations.

1.4.22. Note 1. In general the measure μ which represents a derivation D on $L^1(\omega)$ is not necessarily in $M(\omega)$. For example, for ω_1 as in example 1.4.21, let μ be a measure defined by $d\mu(t) = e^{t^2} dt$. Then μ is not in $M(\omega)$. But,

$$\sup \frac{x}{\omega_1(x)} \int_0^\infty \omega_1(x+y)\, d|\mu|(y) = \sup \frac{x}{e^{-x^2}} \int_0^\infty e^{-x^2-y^2-2xy} e^{y^2} dy$$

$$= \sup \frac{x}{2x} = \frac{1}{2} < \infty .$$

On the other hand, not only for ω_1 but for a general ω, it is not true that every $\mu \in M(\omega)$ gives a derivation as (1) of Theorem 1.4.17. For example, let $\mu \in M(\omega)$, which has non-zero mass at 0, then

$$\sup_{t\in(0,\infty)} \frac{t}{\omega(t)} \int_0^\infty \omega(t+s)\, d|\mu|(s) \geq \sup_{t\in(0,\infty)} \frac{t}{\omega(t)} \omega(t) |\mu| \{0\} = \infty$$

Note 2. We saw that for every derivation D on $L^1(\omega)$, there corresponds a measure μ which satisfies (1) of theorem 1.4.17. This correspondence is one-to-one since if $D = 0$ then the formula for the norm of D in corollary 1.4.18 gives $\mu = 0$. The complete-ness of the space of continuous derivations on a Banach algebra imply that for every weight ω which satisfies the hypothesis of theorem 1.4.20 there is a concrete example of Banach space of measures defined by

$$B_\omega = \{\mu : \sup_{t>0} \frac{t}{\omega(t)} \int_0^\infty \omega(t+s)d|\mu|(s) < \infty\}$$

with norm

$$||\mu|| = \sup \frac{t}{\omega(t)} \int_0^\infty \omega(t+s)d|\mu|(s) < \infty \qquad (\mu \in B_\omega)$$

Note 3. Wermer and Singer [cf.35] have shown that in a semi-simple commutative Banach algebra there exist no non-trivial continuous derivations. Wermer has conjectured the following converse. If a commutative Banach algebra has no non-trivial derivations then it is semi-simple. D.J. Newman has given a counter example to this conjecture [cf.25]. The algebra $L^1(\omega_3)$ with ω_3 as in 1.4.18 is another counter example to this conjecture.

By using the methods of [20], with minor changes, for every algebra $L^1(\omega)$ with non-zero derivations we find when two derivations D_1 and D_2 on $L^1(\omega)$ commute. We use the larger algebra $L^1{}_{loc}$ [see definition 1.4.3] and together with it the algebra M_{loc} which consists of all Borel measures with finite variations on every compacta, where the addition of two measures and scalar multiplication are defined as follows: if E is a bounded Borel set then,

$$(1) \qquad (\mu + \nu)(E) = \mu(E) + \nu(E) \qquad (\mu, \nu \in M_{loc})$$

$$(\alpha\mu)(E) = \alpha\mu(E) \qquad (\mu \in M_{loc}, \alpha \in \underset{\sim}{C})$$

The product in M_{loc} is defined as follows. For μ and ν in M_{loc} let $\mu*\nu$ be a measure in M_{loc} such that $\mu*\nu$ restricted to $[0, n]$ satisfies

$$\int_0^n \psi(x)d(\mu*\nu)(x) = \int_0^n \int_0^{n-y} \psi(x+y)d\mu(x)d\nu(y) \qquad (\psi \in C[0, n])$$

The algebra $L^1{}_{loc}$ can be regarded as a subalgebra of M_{loc} in the usual way: for $f \in L_{loc}$, let μ be a measure in M_{loc} such that

$$(d\mu)(x) = f(x)dx$$

In fact $L^1{}_{loc}$ is an ideal in M_{loc}, if $f \in L_{loc}$, $\mu \in M_{loc}$, then

$$\int_0^n \psi(x)d(\mu*f)(x) = \int_0^n \int_0^{n-y} \psi(x+y)f(x)dx\, d\mu(y)$$

$$= \int_0^n \int_y^n \psi(z)f(z-y)dz\, d\mu(y) = \int_0^n \psi(z) \int_0^y f(z-y)\, d\mu(y)$$

Thus $\mu*f$ corresponds to the function h defined by

$$h(y) = \int_0^y f(x-y)\,d\mu(y) \qquad (y \in R^+)$$

which is in L^1_{loc}. The algebras $L^1(\omega)$ and $M(\omega)$, can respectively be regarded as subalgebras of L^1_{loc} and M_{loc}. In 1.4.4 we saw that L^1_{loc} is an integral domain. This is true for M_{loc} as well.

1.4.23 Lemma. M_{loc} is an integral domain.

Proof. If $\mu, \nu \in L^1_{loc}$ with $\mu*\nu = 0$, and $\mu*g \neq 0$ for some $g \in L^1_{loc}$ then for every $h \in L^1_{loc}$ we have,

$$(g*\mu) * (\nu*h) = 0$$

Since $g*\mu$, $\nu*h \in L^1_{loc}$, Titchmarsh's convolution theorem implies $\nu*h = 0$ for all $h \in L^1_{loc}$. Thus, without loss of generality we can assume that $\mu*g = 0$, for every $g \in L^1_{loc}$. Thus,

$$(1) \qquad \int_0^x g(x-y)\,d\mu(y) = 0 \qquad (x \in R^+)$$

Now, if E is a Borel set, for every positive integer n let $g = \chi_{E\cap[0,n]}$. If $x > n$, then from (1) we obtain $\mu(E\cap[0,n]) = 0$. Thus, $\mu(E\cap(n, n+1]) = \mu((E\cap[0, n+1])\setminus(E\cap[0, n])) = 0$. Therefore,

$$\mu(E) = \mu(E\cap[0,1]) + \sum_{n=1}^{\infty} \mu(E\cap(n, n+1]) = 0$$

1.4.24 Lemma. Let D_1 and D_2 be two derivations on $L^1(\omega)$ with $D_i f = tf*\mu_i$ $(i = 1, 2)$, $\left(f \in L^1(\omega)\right)$. Then $D_1D_2 = D_2D_1$ if and only if $\mu_1*x\,\mu_2 = x\mu_1 * \mu_2$.

Proof. If $D_1D_2 = D_2D_1$ then for $f \in L^1(\omega)$,

$$D_1D_2f = x(xf*\mu_2)*\mu_1 = x^2f*\mu_2*\mu_1 + xf*x\mu_2*\mu_1$$

and

$$D_2D_1f = x(xf*\mu_1)*\mu_2 = x^2f*\mu_1*\mu_2 + xf*x\mu_1*\mu_2$$

Thus $D_1D_2 = D_2D_1$ if and only if $xf*(x\mu_1*\mu_2) = xf*(\mu_1*x\mu_2)$ for all f. Thus, by lemma 1.4.23, $D_1D_2 = D_2D_1$ if and only if $x\mu_1*\mu_2 = \mu_1*x\mu_2$.

Lemma 1.4.25 Let $f, g \in L^1_{loc}$ and suppose that $xf*g = f*xg$. Then $x^nf*g = f*x^ng$ for all positive integers n.

Proof. Suppose $x^nf*g = f*x^ng$ for some $n \geq 1$. Convolving with g gives $x^nf*g*g = f*x^ng*g$. Multiplying by x and using the fact that multiplication by x is a derivation on L^1_{loc} we get $x^{n+1}f*g*g + x^nf*xg*g + x^nf*g*xg = xf*x^ng*g + f*x^{n+1}g*g + f*x^ng*xg$. Using commutativity of $*$ and the hypothesis we obtain,

$$x^{n+1}f*g*g + 2x^nf*g*xg = 2f*x^ng*xg + f*x^{n+1}g*g \quad .$$

Using the induction hypothesis we obtain $x^{n+1}f*g = f*x^{n+1}g$ and this completes the induction.

Lemma 1.4.26 Let g be continuous on R^+. Then the only continuous solutions to $xf*g = f*xg$ are of the form, $f = cg$, where c is a constant.

Proof. It is obvious that $f = cg$ satisfies $xf*g = f*xg$. Conversely, suppose $xf*g = f*xg$. Then by lemma 1.4.25 we have that $x^nf*g = f*x^ng$ for all n and hence $Pf*g = f*Pg$ for polynomials P. Now let b be the inf $s(g)$ and let x be a fixed number bigger than b, then we have

$$(1) \quad \int_0^x P(t)f(t)g(x-t)dt = \int_0^x f(x-t)P(t)g(t)dt$$

This equation is also true if we replace P by a bounded measurable function on $[0, x]$. For every $a \le x$ let

$$P_a(t) = \begin{cases} 1 & 0 \le t \le a \\ 0 & \text{elsewhere} \end{cases}$$

Then (1) becomes

$$(2) \qquad \int_0^a f(t)g(x-t)\,dt = \int_0^a f(x-t)g(t)\,dt$$

This holds for all $a \le x$. Differentiating with respect to a we obtain $f(a)g(x-a) = f(x-a)g(a)$ for all x and all $a \le x$. If we now let $a \to b^+$ through values for which $g(a) \ne 0$ we obtain $f(x-b) = cg(x-b)$, with c the common value of $\frac{f(a)}{g(a)}$. That is $f(t) = cg(t)$ $(t \in R^+)$.

1.4.27 Theorem. Let D_1 and D_2 be two non-zero derivations on $L^1(\omega)$. Then $D_1D_2 = D_2D_1$ if and only if $D_2 = cD_1$, where c is a constant.

Proof. By lemma 1.4.24 $D_1D_2 = D_2D_1$ if and only if $x\mu_1*\mu_2 = \mu_1*x\mu_2$. This holds if and only if $x*x\mu_1*\mu_2*x = x*\mu_1*x\mu_2*x$, or equivalently

$$x*x\mu_1*\mu_2*x + x^2*\mu_1*x*\mu_2 = x*\mu_1*x\mu_2*x + x^2*\mu_1*x*\mu_2$$

which is

$$(x*x\mu_1 + x^2*\mu_1)*(x*\mu_2) = (x*\mu_1)*(x\mu_1*x + x^2*\mu_2)$$

which is

$$x(x*\mu_1)*(x*\mu_2) = (x*\mu_1)*(x\mu_2*x + x^2*\mu_2)$$

which is

$$x(x*\mu_1)*(x*\mu_2) = (x*\mu_1)*x(x*\mu_2) .$$

Repeating the argument with $x*\mu_i$ replacing μ_i we obtain that

$D_1D_2 = D_2D_1$ if and only if

$$x(x*x*\mu_1)*(x*x*\mu_2) = (x*x*\mu_1)*x(x*x*\mu_2)$$

Now $x*x*\mu_i$ is continuous on R^+. By lemma 1.4.26, $D_1D_2 = D_2D_1$ if and only if $x*x*\mu_1 = cx*x*\mu_2$, by lemma 1.4.23, this is equivalent to $\mu_1 = c\mu_2$ on R^+ or $D_1 = cD_2$.

Given a pair of radical weights ω_1 and ω_2 we find necessary and sufficient conditions under which $L^1(\omega_2)$ is a two-sided Banach $L^1(\omega_1)$-module under the module product,

$$(f,\ g) \to f*g \qquad (f \in L^1(\omega_1),\ g \in L^1(\omega_2))$$

Having found these necessary and sufficient conditions we characterize all derivations from $L^1(\omega_1)$ into $L^1(\omega_2)$.

1.4.28 Lemma. A necessary and sufficient condition for $L^1(\omega_2)$ to be a two-sided Banach $L^1(\omega_1)$ - module under the module product $*$ is that,

$$\sup_{t\in R^+} \frac{\omega_2(t)}{\omega_1(t)} = K < \infty .$$

Proof. If

$$\sup_{t\in R^+} \frac{\omega_2(t)}{\omega_1(t)} < \infty$$

We show that if $f \in L^1(\omega_1)$, $g \in L^1(\omega_2)$ then $f*g \in L^1(\omega_2)$ and there exists a constant $M > 0$ such that $\|f*g\| \le M\|f\|\,\|g\|$.

We have,

$$(1)\quad \int_0^\infty |(f*g)(x)|\omega_2(x)\,dx = \int_0^\infty\left|\int_0^x f(x-y)g(y)\,dy\right|\omega_2(x)\,dx \le \int_0^\infty\int_0^x |f(x-y)|\,|g(y)|\,dy\,\omega_2(x$$

The last integral of (1) by Fubini's theorem is equal to

$$(2)\quad \int_0^\infty |g(y)|\int_0^\infty |f(x)|\omega_2(x+y)\,dx\,dy \le K\int_0^\infty |g(y)|\omega_2(y)\,dy \int_0^\infty |f(x)|\omega_1(x)\,dx$$

$$= K\,\|f\|\,\|g\| .$$

Conversely, if $L^1(\omega_2)$ is a two-sided Banach $L^1(\omega_1)$ - module then there exists a positive real number K such that for every $f \in L^1(\omega_1)$ and $g \in L^1(\omega_2)$

$$(3) \qquad \int_0^\infty \left|\int_0^x g(x-y)f(y)\,dy\right| \omega_2(x)\,dx \le K \,\|f\|\,\|g\|$$

Now, for $f \in L^1(\omega_1)$ and $g \in L^1(\omega_2)$ let

$$\phi(f, g) = \int_0^\infty \int_0^x g(x-y)f(y)\,dy\ \omega_2(x)\,dx$$

By Fubini's theorem

$$\phi(f, g) = \int_0^\infty f(y) \int_y^\infty g(x-y)\omega_2(x)\,dxdy$$

$$= \int_0^\infty f(y) \int_0^\infty g(x)\ \omega_2(x+y)\,dxdy$$

For a fixed $g \in L^1(\omega_2)$, the map

$$f \to \phi(f, g) \qquad (f \in L^1(\omega_1))$$

is a linear functional on $L^1(\omega_1)$ which is continuous by (3). Thus

$$(4) \qquad \frac{\left|\int_0^\infty g(x)\omega_2(x+y)\,dx\right|}{\omega_1(y)} \le K\,\|g\| \qquad (g \in L^1(\omega_2),\ a.e.\ y \in R^+)$$

For every $y \in R^+$, the map

$$g \to \frac{\int_0^\infty g(x)\omega_2(x+y)\,dx}{\omega_1(y)}$$

is by (4) a continuous linear functional on $L^1(\omega_2)$. Thus

$$\frac{\omega_2(x+y)}{\omega_1(y)\omega_2(x)} \le K \qquad (a.e.\ x \in R^+, a.e.\ y \in R^+)$$

and hence for all $x, y \in R^+$ by continuity of ω_1 and ω_2.
In particular for $x = 0$, we get

$$\sup_{y \in R^+} \frac{\omega_2(y)}{\omega_1(y)} \le K .$$

and this proves the lemma.

Given a derivation D from $L^1(\omega_1)$ into $L^1(\omega_2)$ we can extend it to a derivation $\overline{D}$ from $M(\omega_1)$ into $M(\omega_2)$ and then characterize $\overline{D}$ and D and find necessary and sufficient conditions on ω_1 and ω_2 for the existence of non-zero derivations. The arguments are similar to those of the derivations of a single algebra and we only state the results.

1.4.29 Theorem. If D is a derivative from $L^1(\omega_1)$ into $L^1(\omega_2)$ then there is a locally finite measure μ such that,

$$||D|| = \sup_{y>0} \frac{y}{\omega_1(y)} \int_0^\infty \omega_2(y+s)\, d|\mu|(s)$$

and

$$Df = tf*\mu \qquad (f \in L^1(\omega_1)) .$$

A necessary and sufficient condition for the existence of a non-zero derivation is the existence of positive real number b such that

$$\sup_{a>0} \frac{a}{\omega_1(a)} \omega_2(a+b) < \infty .$$

The result of this theorem leads to a characterization of the first cohomology group of $L^1(\omega_1)$ with coefficients in $L^1(\omega_2)$, $H^1(L^1(\omega_1), L^1(\omega_2))$ (for the definition of $H^1(A, X)$ when X is a two-sided Banach A-module see [3], p.238).

PART TWO

CHAPTER 2.1

In this part we show that there is an isometric isomorphism from $M(G)$ into $BB(H)$, where H is a Hilbert space, $B(H)$ is the algebra of bounded operators on H and $BB(H)$ is the algebra of bounded operators on $B(H)$. As a corollary we deduce that $L^1(G)$ has an isometric representation in $BB(H)$. We also show that $L^1(G)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space.

2.1.1 Definition. Let G be any group, E any non-void set, and f any function with domain G and range E. For a fixed element $a \in G$ let $L_a f [R_a f]$ be the function on G such that $(L_a f)(x) = f(ax)$ $[(R_a f)(x) = f(xa)]$ for all $x \in G$. Then $L_a f$ $[R_a f]$ is called the left translate [right translate] of f by a.

2.1.2 Notation. For every $x \in G$ and $A \subset G$, let $xA = \{xa : a \in A\}$ and $Ax = \{ax : a \in A\}$.

2.1.3 Definition. Let G be a group and let F be a family of subsets of G. Let E be any non-void set, and let λ be a function with domain F and range contained in E. Suppose that $A \in F$ and $x \in G$ imply $xA \in F$ $[Ax \in F]$. If $\lambda(xA) = \lambda(A)$ for all $x \in G$ and $A \in F$ $[\lambda(Ax) = \lambda(A)$ for all $x \in G$ and $A \in F]$, then λ is said to be left invariant [right invariant].

2.1.4 Definition. Let G be a set that is a group and also a topological space, we call G a topological group if,

(I) The mapping $(x, y) \to xy$ of $G \times G$ into G is continuous.

(II) The mapping $x \to x^{-1}$ of G onto G is continuous.
We denote the identity element of G by e.

The topological groups with which we will be concerned will all be locally compact and Hausdorff (T_2) groups. For every topological group G there exists a non-negative measure λ defined on the σ-algebra of Borel sets, such that

(I) $\lambda(F) < \infty$, if F is compact;

(II) $\lambda(U) > 0$, for some open set U;

(III) $\lambda(aB) = \lambda(B)$ for B a Borel subset of G and $a \in G$.

[λ is a left invariant in the sense of 2.1.3].

(IV) λ is a regular measure.

Moreover, for every non-negative Borel measure μ which satisfies (I) - (IV) there is a positive constant c, such that $\mu = c\lambda$. For the existence and uniqueness of λ see [16 p.194]. The measure λ is the so called left Haar measure of G. The measure λ which satisfies (I) - (IV) has also the following property,

(V) $\lambda(U) > 0$ for every non void open set U.

For if $\lambda(U) = 0$ and K is compact, finitely many translates of U cover K, and hence $\lambda(K) = 0$. The regularity of λ then implies that $\lambda(B) = 0$ for all Borel sets B in G, a contradiction.

We fix the left Haar measure of a group.

2.1.5 Theorem. Let G be a locally compact group $f \in C_c^+(G)$ $f \neq 0$, and for $x \in G$, let

$$\Delta(x) = \frac{\int_G f(yx^{-1})d\lambda(y)}{\int_G f(y)d\lambda(y)} \quad .$$

Then Δ depends only upon x, and not upon f. The function Δ is continuous, positive throughout G, and satisfies the functional equation

$$\Delta(xy) = \Delta(x)\Delta(y) \quad \text{for all} \quad x, y \in G$$

Proof. [cf.16, Th.15.11, p.195].

The function Δ is called the modular function of the locally compact group G.

2.1.6 Theorem. Let f be a λ-integrable function on G, then for every $a \in G$, the functions $L_a f$ and $R_a f$ are λ-integrable and we have

(I) $$\int_G (L_a f)(x)d\lambda(x) = \int_G f(x)d\lambda(x)$$

(II) $$\int_G (R_a f)(x)d\lambda(x) = \Delta(a^{-1}) \int_G f(x)d\lambda(x)$$

Proof. [cf.10, Th.20.1, p.283].

2.1.7 Notation. We let $L^p(G)$ be the Banach space $L^p(G, \lambda)$ $(1 \leq p \leq \infty)$. The space $L^2(G)$ with the inner product

$$\langle f, g\rangle = \int_G f(x)\, \overline{g}(x)\, d\lambda(x)$$

is a Hilbert space. From 2.1.6 it follows that L_a, and R_a are bounded operators on $L^p(G)$.

2.1.8 Lemma. For $s, t \in G$, we have $L_s L_t = L_{ts}$, and L_t is a unitary operator on $L^2(G)$ with $(L_t)^* = L_{t^{-1}}$.

Proof. Given $f \in L^2(G)$ we have,

$$(L_t L_s f)(x) = (L_s f)(tx) = f(stx) = (L_{st} f)(x) \qquad (x \in G)$$

Thus, $L_t L_s = L_{st}$. In particular $L_{t^{-1}} L_t = L_{tt^{-1}} = L_e = 1$.

This together with 2.1.6 (I) show that L_t is a unitary operator and $(L_t)^* = L_{t^{-1}}$.

2.1.9 Theorem. Let $1 \leq p < \infty$, and let f be a function in $L^p(G)$. For every $\varepsilon > 0$, there is a neighbourhood U of e in G such that

$$\|L_s f - L_t f\| < \varepsilon \quad \text{if} \quad s, t \in G \quad \text{and} \quad st^{-1} \in U .$$

That is, the mapping $x \to L_x f$ of G into $L^p(G)$ is right uniformly continuous.

Proof. [cf.16, Th.20.4, p.285].

CHAPTER 2.2

The Algebras $L^1(G)$ and $M(G)$

2.2.1 Definition. Given a locally compact group G, let λ be the left Haar measure on G, then $L^1(G)$ becomes a Banach algebra with the product given by convolution,

$$(f*g)(s) = \int_G f(t)g(t^{-1}s)\, d\lambda(t) \qquad (s \in G)$$

Let $M(G)$ denote the Banach space of all finite complex regular Borel measures on G, with usual addition of measures and scalar multiplication and norm defined by

$$(1) \qquad (\mu + \nu)(E) = \mu(E) + \nu(E) \qquad (\mu, \nu \in M(G), E \in B)$$

$$(2) \qquad (\lambda\mu)(E) = \lambda\mu(E) \qquad (\lambda \in \underset{\sim}{C}, E \in B)$$

$$(3) \qquad \|\mu\| = \sup \sum |\mu(E_i)|$$

where sup in (3) extends over all possible disjoint partitions of G into measurable sets. Let $C_0(G)$ be the Banach space of continuous complex valued functions on G which vanish at infinity, with the uniform norm. Then we can identify $M(G)$ with the dual of $C_0(G)$ by the following pairing,

$$\langle \mu, f \rangle = \int_G f(x)\,d\mu(x) \qquad (f \in C_0(G), \mu \in M(G))$$

For $\mu, \nu \in M(G)$, the mapping $f \to \int_G \int_G f(xy)\,d\mu(x)\,d\nu(y)$ is a bounded linear functional on $M(G)$, let $\mu*\nu$ be a measure on $M(G)$ which satisfies the following equation,

$$\int_G f(x)\,d(\mu*\nu)(x) = \int_G \int_G f(xy)\,d\mu(x)\,d\nu(y) \qquad (f \in C_0(G))$$

The algebra $M(G)$ with product $*$ is a Banach algebra.

2.2.2 Definition. A measure $\mu \in M(G)$ is said to be absolutely continuous if μ is absolutely continuous with respect to the left Haar measure. We denote the set of all absolutely continuous measures by $M_a(G)$.

Given $\mu \in M_a(G)$, by the Radon-Nikodym theorem, there is $f \in L^1(G)$ such that $d\mu(x) = f(x)d\lambda(x)$ and $||\mu|| = \int_G |f(x)|d\lambda(x)$. Conversely if $f \in L^1(G)$, the measure $d\mu_f = fd\lambda$ is absolutely continuous. We have the following result.

2.2.3 Theorem. The set $M_a(G)$ is a closed two-sided ideal in the algebra $M(G)$. The map $f \to \mu_f$ from $L^1(G)$ into $M_a(G)$ is an isometric isomorphism of $L^1(G)$ onto $M_a(G)$.

Proof. [cf.10, Th.19.18, p.272].

2.2.4 Theorem. The algebra $L^1(G)$ contains a bounded approximate identity $\{f_\lambda : \lambda \in \Lambda\}$ with $||f_\lambda|| = 1$.

Proof. [cf.16, Th.20.27, p.303].

2.2.5 Theorem. Let $f \in L^1(G)$, then the function g defined by $g(x) = f(x^{-1})$ $(x \in G)$ is in $L^1(G)$ and

$$\int_G g(x)d\lambda(x) = \int_G f(x)\ \Delta(x^{-1})d\lambda(x)$$

$$\int_G f(x)d\lambda(x) = \int_G g(x)\ \Delta(x^{-1})d\lambda(x)$$

Proof. [cf.16, Th.20.2, p.284].

2.2.6 Corollary. The map $f \to f^*$, where

$$f^*(x) = \Delta(x^{-1})\ \overline{f(x^{-1})} \qquad (x \in G)$$

is an involution on $L^1(G)$.

Proof. This follows from 2.2.5 and multiplicativity of the function Δ .

To demonstrate an isometric representation for $L^1(G)$ first we need the following lemma.

2.2.7 Lemma. Let $F_1, F_2, \ldots F_n$ $(n \geq 2)$ be n disjoint compact subsets of G , then there is an open neighbourhood A of e such that for every $x \in F_i$, $y \in F_j$ $(i \neq j)$ $xA \cap yA = \emptyset$ $(i, j = 1, 2, \ldots n)$.

Proof. Since G is a Hausdorff space, for a fixed $x \in F_1$ and every $y \in F_2$ there are two disjoint open sets $O_1(x, y)$ and $O_2(x, y)$ with $x \in O_1(x, y)$ and $y \in O_2(x, y)$. The family $\{O_2(x, y) : y \in F_2\}$ is a cover for F_2 , thus it has a finite subcover $O_2(x, y_1)$, ..., $O_2(x, y_n)$. The two open sets $O_1 = \bigcap_{i=1}^{n} O_1(x,y_i)$ and $O_2 = \bigcup_{i=1}^{n} O_2(x, y_i)$ separate x and F_2 . By compactness of F_1 and by a similar argument we can separate F_1 and F_2 by two open sets N_1 and N_2 and by induction we can separate $F_1, F_2, \ldots, F_n$ by open sets $N_1, \ldots, N_n$. Now let f be the map from $G \times G \to G$ defined by $f(x, y) = xy$, then f is continuous, and for every $x \in F_1$, we have $f(e, x) = x$. Since N_1 is a neighbourhood of x , there is an open set $A(e, x)$ containing e , and an open set $B(x)$ containing x , such that $B(x) \times A(e, x) \subset f^{-1}(N_1)$. Again, the family $\{B(x) : x \in F_1\}$ is a cover for F_1 , thus there are $x_1, x_2, \ldots, x_r \in F_1$ with $\bigcup_{k=1}^{r} B(x_k) \supset F_1$. Let $A_1 = \bigcap_{k=1}^{r} A(e, x_k)$, then $F_1 \times A_1 \subset N_1$ or equivalently $xA_1 \subset N_1$ $(x \in F_1)$. Similarly let A_i $(i = 1, 2, \ldots, n)$ be such that $xA_i \subset N_i$ $(x \in F_i)$, $(i = 1, 2, \ldots, n)$

Now the set $A = \bigcap_{i=1}^{n} A_i$ has the property in the statement of our lemma.

2.2.8 Definition. A sesquilinear form on a Hilbert space H is a mapping $\psi : H \times H \to \mathbb{C}$ such that $\psi(x, y)$ is linear with respect to x and conjugate linear with respect to y.

If ψ is a sesquilinear form on a Hilbert space H and bounded in the sense that $\sup\{|\psi(x, y)| : \|x\| = \|y\| = 1\} = M < \infty$ then there is a bounded linear operator T on H, such that

$$\psi(x, y) = \langle Tx, y\rangle \qquad (x, y \in H)$$

moreover $\|T\| = M$. [cf.28, Th.12.8, p.296].

From now on, unless otherwise stated, we assume $H = L^2(G)$.

2.2.9 Lemma. Let $\mu \in M(G)$ be a non negative measure, $T \in B(H)$ and $f \in L^1(G, \mu)$, then the mapping

$$(1) \qquad (g, h) \to \int_G f(\alpha) \langle L_{\alpha^{-1}} T L_\alpha g, h\rangle \, d\mu(\alpha)$$

defines a bounded sesquilinear form on H.

Proof. First we prove that the above integral exists. Since the map $\alpha \to L_{\alpha^{-1}}$ from G into $B(H)$ is strongly continuous (by 2.1.9) the map $\alpha \to \langle L_{\alpha^{-1}} T L_\alpha g, h\rangle$ is continuous, moreover, since each L_α is a unitary operator we have $|\langle L_{\alpha^{-1}} T L_\alpha g, h\rangle| \le \|T\| \, \|g\| \, \|h\|$ thus the function $\alpha \to f(\alpha) \langle L_{\alpha^{-1}} T L_\alpha g, h\rangle$ is μ-integrable, and the integral exists, an easy computation shows that

$$\left|\int_G f(\alpha) \langle L_{\alpha^{-1}} T L_\alpha g, h\rangle \, d\mu(\alpha)\right| \le \|f\| \quad \|T\| \, \|g\| \, \|h\|$$

and the mapping (1) is linear in g and conjugate linear in h.

For every $f \in L^1(G, \mu)$, let $\psi(f)$ be an operator on $B(H)$, such that for $T \in B(H)$, $\psi(f)T$ is the operator corresponding to the form $(g, h) \to \int_G f(\alpha) \langle L_{\alpha^{-1}} T L_\alpha g, h\rangle \, d\mu(\alpha)$. Thus,

$$\langle \psi(f)T g, h\rangle = \int_G f(\alpha) \langle L_{\alpha^{-1}} T L_\alpha g, h\rangle \, d\mu(\alpha)$$

$$(g, h \in H, T \in B(H), f \in L^1(G, \mu))$$

Thus ψ is a map from $L^1(G, \mu)$ into $BB(H)$.

In the next lemma for every $f \in L^1(G, \mu)$ let $\|f\|_{\mu, 1}$ denote the norm of f as an element of $L^1(G,\mu)$.

Lemma 2.2.10 Let μ be a non-negative measure then the map ψ from $L^1(G, \mu)$ into $BB(H)$ defined by

$$\langle \psi(f)T g, h\rangle = \int_G f(\alpha) \langle L_{\alpha^{-1}} T L_\alpha g, h\rangle \, d\mu(\alpha)$$

$$(f \in L^1(G, \mu), T \in B(H), g, h \in H)$$

defines an isometric isomorphism from the Banach space $L^1(G, \mu)$ into $BB(H)$.

Proof. Obviously ψ is a linear map, ψ is also continuous since

$$\|\psi(f)T\| = \sup\{|\int_G f(\alpha) \langle L_{\alpha^{-1}} T L_\alpha g, h\rangle \, d\mu(\alpha)| : \|g\| = \|h\| = 1\}$$

$$\leq \int_G |f(\alpha)| d\mu(\alpha) \ \|T\| = \|f\|_{\mu,1} \|T\| .$$

Thus ψ is continuous. To prove ψ is an isometry first let $f \in L^1(G, \mu)$ be a simple function $f = \sum_{i=1}^{n} c_k \chi_{F_k}$, where F_k $(k = 1, 2, \ldots n)$ are disjoint compact sets, then $\|f\|_{\mu,1} = \sum_{k=1}^{n} |c_k| \mu(F_k)$. We also let $c_k = |c_k| e^{i\theta_k}$ $(k = 1, 2, \ldots n)$

be the polar form of the number $c_k (k = 1, 2, \ldots, n)$. For the compact sets $F_1, F_2, \ldots, F_n$ we choose the open set A as in lemma 2.2.9, since A is open we have $0 < \lambda(A)$, moreover we can choose A as small as $\lambda(A) < \infty$. Now let $g = \chi_A$, then $g \in L^2(G)$ and $g \neq 0$. Let $M = \text{linear span } \{L_\alpha g : \alpha \in \bigcup_{i=1}^{n} F_i\}$. If $\alpha \in F_i$, $\beta \in F_j$ $(i \neq j)$ the two sets $\chi_{\alpha A}$ and $\chi_{\beta A}$ are disjoint, thus the two functions $L_\alpha g = \chi_{\alpha A}$ and $L_\beta g = \chi_{\beta A}$ are orthogonal. We define the operator S on M as follows, if $f = \sum_{p,q} \lambda_{p,q} L_{\alpha_{p,q}} g$ with $\alpha_{p,q} \in F_p$ $(p = 1, 2, \ldots, n)$ then

$$Sf = \sum_{p,q} e^{-i\theta_p} \lambda_{p,q} L_{\alpha_{p,q}} g$$

S is obviously linear and

$$\|f\|^2 = \sum_{p=1}^{n} \|\Sigma_q \lambda_{p,q} L_{\alpha_{p,q}} g\|^2 = \|Sf\|^2$$

Thus, S is an isometry. We extend S to the closure $\overline{M}$ of M by continuity and we let $T = \overline{S} \oplus 1$ act on $\overline{M} \oplus (\overline{M})^{\perp} = H$, obviously T is an isometry and

$$\langle \psi(f)Tg, g\rangle = \int_G f(\alpha) \quad \langle L_{\alpha^{-1}} TL_\alpha g, g\rangle \, d\mu(\alpha)$$

$$= \sum_{k=1}^{n} c_k \int_{F_k} \langle L_{\alpha^{-1}} TL_\alpha g, g\rangle \, d\mu(\alpha)$$

$$= \sum_{k=1}^{n} c_k \int_{F_k} \langle e^{-i\theta k} L_{\alpha^{-1}} L_\alpha g, g\rangle \, d\mu(\alpha)$$

$$= \sum_{k=1}^{n} c_k e^{-i\theta_k} \mu(F_k) \cdot \|g\|^2 = \sum_{k=1}^{n} |c_k| \mu(F_k) \; \|g\|^2$$

Thus, $\|\psi(f)\| = \|f\|_{\mu, 1}$.

For a general simple function $f = \sum_{k=1}^{n} c_k \chi_{F_k}$ we can by regularity of the measure μ find pairwise disjoint compact sets $F'_k \subset F_k$ such that $\mu(F_k) - \mu(F'_k)$ is arbitrarily small. Now, for the function $f' = \Sigma\, c_k \chi_{F'_k}$ we have $\|\psi(f')\| = \|f'\|_{\mu,1}$ and the continuity of ψ implies $\|f\|_{\mu,1} = \|\psi(f)\|$. Finally, since simple functions are dense in $L^1(G,\mu)$ the continuity of ψ implies that $\|\psi(f)\| = \|f\|_{\mu,1}$. Thus ψ is an isometry.

2.2.11 Theorem. There exists an isometric isomorphism from the algebra $M(G)$ into $BB(H)$.

Proof. We define the map $\theta : M(G) \to BB(H)$, by

$$(1)\qquad \langle\theta(\mu)Tg,h\rangle = \int_G \langle L_{\alpha^{-1}} T L_\alpha g,h\rangle \, d\mu(\alpha)$$

$$(\mu \in M(G),\ T \in B(H),\ g,\ h \in H)$$

Obviously, θ is linear. By the Radon-Nikodym theorem there is a Borel measurable function k with $|k(x)| = 1,\ (x \in G)$ and $d\mu = k\,d|\mu|$. Thus

$$(2)\qquad \langle\theta(\mu)Tg,h\rangle = \int_G k(\alpha)\ \langle L_{\alpha^{-1}} T L_\alpha g,h\rangle \, d|\mu|(\alpha) .$$

Now, let ψ be the mapping of $L^1(G,|\mu|)$ into $BB(H)$ as in lemma 2.2.10, then $\theta(\mu) = \psi(k)$, and by lemma 2.2.5,

$$\|\theta(\mu)\| = \|\psi(k)\| = \|k\|_{|\mu|,1} = \int_G |k(x)|\, d|\mu|(x) = \|\mu\| .$$

Thus, θ is isometric. Given $\mu,\ \nu \in M(G)$, we have

$$\langle\theta(\mu)\ \theta(\nu)T\ g,h\rangle = \int_G \langle L_{\alpha^{-1}}\theta(\nu)TL_\alpha g,h\rangle\, d\mu(\alpha) = \int_G \langle\theta(\nu)TL_\alpha g,\ L_\alpha h\rangle d\mu(\alpha)$$

$$= \int_G\int_G \langle L_{\beta^{-1}}TL_\beta L_\alpha g,\ L_\alpha h\rangle\ d\nu(\beta)d\mu(\alpha)$$

$$= \int_G\int_G \langle L_{\alpha^{-1}}L_{\beta^{-1}}TL_\beta L_\alpha g,h\rangle\ d\nu(\beta)d\mu(\alpha)$$

$$= \int_G\int_G \langle L_{(\alpha\beta)^{-1}}TL_{\alpha\beta}g,h\rangle\ d\nu(\beta)d\mu(\alpha)$$

$$= \int_G \langle L_{\gamma^{-1}}TL_\gamma g,h\rangle\ d(\mu*\nu)(\gamma) = \langle\theta(\mu*\nu)Tg,h\rangle$$

Thus $\theta(\mu)\,\theta(\nu) = \theta(\mu * \nu)$ and θ is an isometric isomorphism of $M(G)$ into $BB(H)$. As a corollary we have

2.2.12 Corollary. There is an isometric isomorphism from $L^1(G)$ into $BB(H)$.

Proof. This follows, since $L^1(G)$ is a subalgebra of $M(G)$. In what follows we assume H is an arbitrary Hilbert space.

Now, we prove that there is no isometric isomorphism from $M(G)$ into $B(H)$, and for this it suffices that we prove there is no isometric isomorphism from $L^1(G)$ into $B(H)$. First we prove that $L^1(G)$ is not isometrically isomorphic with a C*-algebra. We use the techniques of the theory of numerical ranges.

Definition 2.2.13 Let A be a unital Banach algebra with identity 1 , and dual A^* , the numerical range of an element $a \in A$ is a subset $V(a)$ of $\underset{\sim}{C}$ given by

$$V(a) = \{f(a) : f \in A^*, \|f\| = f(1) = 1\}$$

An element $h \in A$ is said to be Hermitian if $V(a) \subset \underset{\sim}{R}$, we denote the set of all Hermitian elements of A by $Her(A)$.

2.2.14 Lemma. Given $h \in Her(A)$ the following statements are equivalent.

(I) $h \in Her(A)$

(II) $\lim\limits_{\alpha \to \infty} \frac{1}{\alpha}\{\|1 + i\alpha h\| - 1\} = 0 \qquad (\alpha \in R)$

(III) $\|\exp(i\alpha h)\| = 1 \qquad (\alpha \in R)$

Proof. [cf.4, Lemma 4, p.46].

2.2.15 Lemma. If $A = Her(A) + i\,Her(A)$ then the map $*$, which to every $x = h + ik$ $(h, k \in H(A))$ associates the element $x^* = h - ik$ is a continuous linear involution on A .

Proof. [cf.4, Lemma 8, p.50].

2.2.16 Theorem [Vidav-Palmer]. Let A be a complex unital Banach algebra then $A = \mathrm{Her}\,A + i\,\mathrm{Her}\,A$ if and only if A is isometrically star isomorphic with a C*-algebra.

Proof. [cf.4, Th.9, p.65].

2.2.17 Lemma. $L^1(G)$ is not isometrically isomorphic with a C*-algebra.

Proof. Since the double centralizer of a C*-algebra is a C*-algebra [cf.5] and the double centralizer of $L^1(G)$ is $M(G)$ [cf.42 and 21 Corollary 0.1.1 p.6 and 38] it is enough that we prove $M(G)$ is not isometrically isomorphic with a C*-algebra. To prove this we use Theorem 2.2.16. Let $\mu \in M(G)$ be Hermitian then $\mu = \mu_a + \mu_s$ (by Radon-Nykodym theorem) where μ_a is absolutely continuous and μ_s is singular with respect to δ_e (the Dirac measure at e). The measure μ_a is therefore, concentrated at e and thus $\mu_a = \lambda\delta_e$ for some $\lambda \in \mathbb{C}$. If μ is Hermitian then by lemma 2.2.14 (II)

$$\lim_{\alpha\to 0} \frac{1}{\alpha} \{\|1 + i\alpha\mu\| - 1\} = \lim_{\alpha\to 0} \frac{1}{\alpha} \{\|\delta_e + i\alpha(\lambda\delta_e + \mu_s)\| - 1\}$$

$$= \lim_{\alpha\to 0} \frac{|1 + i\alpha\lambda| + \alpha\|\mu_s\| - 1}{\alpha} = 0$$

This implies $\mu_s = 0$ and $\lambda \in \mathbb{R}$. Thus $\mathrm{Her}(M(G)) + i\,\mathrm{Her}(M(G)) = \mathbb{C}\delta_e$. Hence, by theorem 2.2.16, $M(G)$ is not isometrically isomorphic with a C*-algebra. Therefore $L^1(G)$ is not isometrically isomorphic with a C*-algebra.

To prove that $L^1(G)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space, we need the following definition and two propositions from the theory of unitary group representations.

2.2.18 Definition. Let G be a locally compact topological group, a unitary representation of G on a Hilbert space H is a group homomorphism $t \to U_t$ of G into the group $U(H)$ of unitary operators on H, a unitary representation $G \to U(H)$ is said to be continuous if it is continuous for the given topology on G and the strong operator topology on $U(H)$.

2.2.19 Proposition. Suppose $\theta : L^1(G) \to B(H)$ is algebra representation such that $\|\theta\| \le 1$ and, closed linear span $\{(\theta f)(x) : f \in L^1(G), x \in H\} = H$. Let $\{e_j : j \in J\}$ be any bounded approximate identity in $L^1(G)$. Then

(a) For each $t \in G$, the net of operators $\theta(L_{t^{-1}}(e_j))$ converges strongly to a unitary operator U_t on H.

(b) For all $f \in L^1(G)$, and $x, y \in H$, we have

$$\langle\theta(f)x,y\rangle = \int_G f(t)\ \langle U_t\ x,y\rangle\ dt$$

Proof. [cf.1, Th.69.20 and exercise 69.30].

2.2.20 Proposition. Suppose $t \to U_t$ is a continuous unitary representation of G on a Hiblert space H. Then, there exists a unique $*$-representation $\theta : L^1(G) \to B(H)$ such that

$$\langle\theta(f)x,y\rangle = \int_G f(t)\ \langle U_t x,y\rangle\ dt$$

for all $f \in L^1(G)$ and $x, y \in H$.

Proof. [cf.1, Th.69.21].

2.2.21 Lemma. If θ is an isometric isomorphism of $L^1(G)$ into $B(H)$ then there is a Hilbert space K and an isometric isomorphism ψ from $L^1(G)$ into $B(K)$ such that

$$K = \text{closed linear span } \{\psi(f)x : f \in L^1(G),\ x \in K\}$$

Proof. Let

$$K = \text{closed linear span } \{\theta(f)x : f \in L^1(G),\ x \in H\}$$

then K is invariant under each $\theta(f)$, $(f \in L^1(G))$. Now, let $\psi = \theta|K$, then ψ is a representation of $L^1(G)$ on K. To prove ψ is isometric, first we note that

(1) $K = \text{closed linear span } \{\psi(f)x : f \in L^1(G),\ x \in K\}$

this is because if

$$x = \alpha_1\ \theta(f_1)x_1 + \alpha_2\theta(f_2)x_2 + \ldots + \alpha_k\theta(f_k)x_k \in K$$

and $\{e_\lambda : \lambda \in \Lambda\}$ is a bounded approximate identity for $L^1(G)$ with $\|e_\lambda\| = 1\ (\lambda \in \Lambda)$ then

$$x_\lambda = \alpha_1\theta(f_1)\theta(e_\lambda)x_1 + \ldots + \alpha_k\theta(f_k)\theta(e_\lambda)x_k$$

is in linear span $\{\psi(f)x : x \in K,\ f \in L^1(G)\}$ and $x_\lambda \to x$.

To prove ψ is isometric, we have $\|\psi(f)\| \le \|\theta(f)\| = \|f\|$ $(f \in L^1(G))$. If $f \ne 0$, let $x \in H$ with $\theta(f)x \ne 0$ and let $y_\lambda = \theta(e_\lambda)\ x \in K$ $(\lambda \in \Lambda)$. Since, $\|\psi(f)\theta(e_\lambda)x\| \to \|\theta(f)x\| \ne 0$, there is a subnet of $\|\theta(e_\lambda)x\|$ that remains bounded away from 0. For this subnet we have

$$\frac{\|\psi(f)y_\lambda\|}{\|y_\lambda\|} = \frac{\|\theta(f)\theta(e_\lambda)x\|}{\|\theta(e_\lambda)x\|} = \frac{\|\theta(f*e_\lambda)x\|}{\|\theta(e_\lambda)x\|}$$

$$\ge \frac{\|\theta(f*e_\lambda)x\|}{\|\theta(e_\lambda)\|\,\|x\|} = \frac{\|\theta(f*e_\lambda)x\|}{\|x\|} \to \frac{\|\theta(f)x\|}{\|x\|}$$

Thus, $\|\psi(f)\| = \sup\left\{\frac{\|\psi(f)y\|}{\|y\|} : y \neq 0, \quad y \in K\right\}$

$$\geq \sup\left\{\frac{\|\theta(f)x\|}{\|x\|} : x \neq 0, \quad x \in H\right\} .$$

Thus $\|\psi(f)\| = \|\theta(f)\| = \|f\|$, therefore ψ is isometric.

2.2.22 Theorem. The algebra $L^1(G)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space.

Proof. Suppose that there exists an isometric isomorphism θ from $L^1(G)$ into $B(H)$ for some Hilbert space H . By lemma 2.2.21 we can without loss of genérality assume that, closed linear span $\{\theta(f)x : x \in H, f \in L^1(G)\} = H$. By proposition 2.2.20 corresponding to θ , there is a unitary representation $t \to U_t$ of G into $U(H)$ such that

(1) $\langle\theta(f)x,y\rangle = \int_G f(t) \langle U_t x,y\rangle d\lambda(t) \quad (f \in L^1(G),\ x,\ y \in H)$.

By proposition 2.2.21 corresponding to the representation $t \to U_t$ there is a $*$-representation ϕ of $L^1(G))$ on H such that

(2) $\langle\phi(f)x,y\rangle = \int_G f(t) \langle U_t x,y\rangle d\lambda(t)$

comparing the right hand sides of (1) and (2) we obtain

$$\langle\theta(f)x,y\rangle = \langle\phi(f)x,y\rangle \qquad (f \in L^1(G),\ x,\ y \in H)$$

thus $\theta = \phi$, and since ϕ is a $*$-representation of $L^1(G)$, we conclude θ is a $*$-representation or equivalently $L^1(G)$ is isometrically isomorphic with a C*-algebra and this contradicts lemma 2.2.17. Thus $L^1(G)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space.

Note 1. We can always find an isometric isomorphism from $L^1(G)$

as a Banach space into $B(H)$, for some Hilbert space. In fact, if B is a Banach space, then B has an isometric embedding in $C(X)$ (Banach-Alaoglu theorem) and $C(X)$ being a C*-algebra by Gelfand-Naimark-Segal construction [cf.3, Th.10 p.209] has an isometric representation on a Hilbert space.

Note 2. N.J. Young has proved that when G is an infinite group, $L^1(G)$ is not Arens regular [cf.41]. On the other hand Civin and Yood in [6] have proved that every C*-algebra is Arens regular, and if A is a Banach algebra which is Arens regular, then every closed subalgebra of A is Arens regular. Thus, every operator algebra is Arens regular. Therefore when G is infinite $L^1(G)$ is not isometrically isomorphic with an algebra of operators on a Hilbert space. However when G is finite $L^1(G)$ is Arens regular [cf.41] and the above method is not applicable.

Note 3. The map $t \to L_{t^{-1}}$ is a continuous unitary representation of the group G on the Hilbert space $L^2(G)$. Formula (1) of Theorem 2.2.11 shows that corresponding to this unitary group representation there is an isometric isomorphism of $M(G)$ into $BB(H)$. However in this formula if we replace $L_{t^{-1}}$ by a continuous representation U_t of G we get a homomorphism from $L^1(G)$ into $BB(H)$, but this homomorphism in general is not isometric. For example let G be the circle group, and for every $z \in G$ let U_z be the operator defined on $L^2(G)$ by

$$(U_z f)(x) = zf(x) \qquad (f \in L^2(G),\ x \in G)$$

Then,

$$\langle \theta(\mu)Tf,g\rangle = \int_G \langle U_{t^{-1}} T U_t f,g\rangle \, d\mu(t) = \int_G \langle t^{-1} T t f,g\rangle \, d\mu(t)$$

$$= \int_G \langle Tf,g\rangle \, d\mu(t) = \langle Tf,g\rangle \int_G d\mu(t)$$

Thus

$$||\theta(\mu)|| = |\int_G d\mu(t)| \quad \text{and} \quad \theta \ \text{is not isometric.}$$

Problem 1. Let $*$ be the involution

$$f^*(x) = \Delta(x^{-1})\ \overline{f(x^{-1})} \qquad (f \in L^1(G),\ x \in G)$$

is there a $*$-isometric isomorphism from the Banach space $L^1(G)$ into $B(H)$ for some Hilbert space H.

Problem 2. In [34] Sinclair has proved that the extremal algebra of $[-1, 1]$, $Ea[-1, 1]$ [cf.8] is the quotient of the group algebra of real numbers with the discrete topology $\ell^1(\underset{\sim}{R})$, by the ideal

$$I = \{\lambda \in \ell^1(\underset{\sim}{R}) : \sum_{\alpha\in R} \lambda(\alpha)e^{i\alpha t} = 0, \quad -1 \le t \le 1\}$$

Is there an isometric isomorphic from $E_a[-1, 1]$ into $BB(H)$? More generally, is there an isometric isomorphism of the quotients of $L^1(G)$ by its closed ideals into $BB(H)$?

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